SKEW MEAN CURVATURE FLOW

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ABSTRACT. The skew mean curvature flow or binormal flow, which origins from the vortex filament equation, describes the evolution of a codimension two submanifold along binormal direction. We show that by a generalized Hasimoto transformation, the SMCF is equivalent to a non-linear Schrödinger system. Moreover, we prove the existence of a local solution to the initial value problem of the SMCF of surfaces in Euclidean space $\mathbb{R}^4$. The key ingredient is a uniform Sobolev-type embedding theorem for the second fundamental forms of two dimensional surfaces, which might be of independent interest.

1. INTRODUCTION

The *skew mean curvature flow* (SMCF) or *binormal flow* describes the evolution of a codimension two submanifold along the binormal direction. More precisely, suppose $I \subset [0, +\infty)$ is an interval, Σ is an n dimensional oriented manifold and $(\overline{M}, \bar{g})$ is an $(n+2)$ dimensional oriented Riemannian manifold. Let $F : I \times \Sigma \to \overline{M}$ be a family of immersions from Σ to $\overline{M}$. For each $t \in I$, let $\mathbf{H}(F)$ denote the mean-curvature vector of the immersion $F(t, \cdot) : \Sigma \to \overline{M}$. There exists a natural almost complex structure J on the normal bundle of Σ, which simply rotates a vector in the normal space by $\pi/2$ positively. We call the rotated vector field $J(F)\mathbf{H}(F)$ the skew mean curvature vector(or binormal vector). Then the SMCF is defined by

$$\frac{\partial F}{\partial t} = J(F)\mathbf{H}(F). \tag{1.1}$$

The SMCF defines a geometric Hamiltonian flow for submanifolds [12]. It is well-known that the mean curvature vector field is the gradient of the volume functional. If we regard the space of immersions as an infinitely dimensional manifold, which is endowed with a symplectic structure induced by the Marsden-Weinstein symplectic form [23], then it is not difficult to see that the SMCF is the Hamiltonian flow of the volume functional. See Section 2.1 for a detailed explanation.

The SMCF stems naturally from the research of hydrodynamics and describes the locally induced motion of codimension two singular vortices, i.e. *vortex membranes*. Recall that motion of an inviscid incompressible fluid on a Riemannian manifold is governed by the Euler equation which has a vorticity form

$$\partial_t \xi + L_v \xi = 0. \tag{1.2}$$

Here v is a divergence free vector field, $\xi := \mathrm{curl} v$ is the vorticity field and L_v denotes the Lie derivative along v. An elegant insight of V. I. Arnold [2] is that the vorticity can be viewed as a 2-form by setting $\xi = dv^b$. This point of view has many unexpected outcomes. In particular, in this way the space of vorticities becomes the dual space of the Lie algebra of divergence-free

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vector fields and possesses a natural Poisson structure. Moreover, the Euler equation defines a Hamiltonian evolution on its symplectic leaves.

However, equation (1.2) is non-local since one has to solve $v = \mathrm{curl}^{-1}\xi$ to find the velocity. To overcome this problem, *the localized induction approximation*(LIA) makes the assumption that the velocity is only induced by local vorticities. By applying the Biot-Savart law and neglecting the higher order terms in the Taylor expansion, one obtains the LIA equation for singular supported vortices.

This method was first utilized by Da Rios [6] for one dimensional vortex filament in $\mathbb{R}^3$ in 1906. It was only realized that the LIA works for higher dimensional vortices as well very recently. Shashikanth [26] first found the LIA equation of vortices supported on two dimensional surfaces in $\mathbb{R}^4$. Then Khesin [18] showed that, in Euclidean space of any dimension, the LIA equation of codimension two vortex membranes is exactly the SMCF.

In particular, the classical case of one dimensional SMCF in Euclidean space $\mathbb{R}^3$ is known as the *vortex filament equation*, which describes the self-induced motion of a thin vortex tube in a perfect fluid. Namely, the vortex filament equation for a time-dependent space curve γ has the form

$$\partial_t \gamma = \gamma_s \times \gamma_{ss}, \tag{1.3}$$

where s is the arc-length parameter and $\times$ denote the exterior product in $\mathbb{R}^3$. If we denote the Frenet frame of γ by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and its curvature by k, then equation (1.3) is just the one dimensional SMCF

$$\partial_t \gamma = k\mathbf{b}.$$

The vortex filament equation (1.3) has been extensively studied by both mathematicians and physicians. One of the reasons which makes it so fascinating is that (1.3) is equivalent to the cubic nonlinear Schrödinger equation. Indeed, by the famous Hasimoto transformation [15]

$$\Phi(t,s) = k(t,s)\exp\left(i\int_{s_0}^{s}\tau(t,s')ds'\right),$$

one can verify that equation (1.3) is equivalent to

$$-\mathbf{i}\Phi_t = \Phi_{ss} + \frac{1}{2}|\Phi|^2\Phi. \tag{1.4}$$

It is well-know that (1.4) is integrable system and admits travelling wave solutions, i.e. solitons. This reveals the rich structure of the vortex filament equation.

The global existence of a solution to the vortex filament equation in the weak sense is obtained in [24] by method of regularization. The SMCF of a curve in a general three manifold and a surface in $\mathbb{R}^4$ was studied in [11]. See also [30] for a discussion of SMCF from the view of integrable systems. On the other hand, there are various evolution equations which are closely related to the vortex filament equation. For example, the Gauss map of a vortex filament satisfies the Landau-Lifshitz equation. A successful and important extension along this vein is the famous *Schrödinger flow* or *Schrödinger map*, which is has been extensively studied in the last decade, see for example [7, 8, 31, 5]. It is also worth mentioning that the singular set of the Gross-Pitaevsky equation evolves along the SMCF [17].

The celebrated *mean curvature flow*(MCF), which is the gradient flow of the volume functional, has played a central role in the field of geometric analysis. Although it seems that there is only a small difference between the MCF and SMCF, the SMCF turns out to be much more difficult to work with than the MCF. From the perspective of partial differential equations, this is because the MCF is a (degenerate) parabolic system while the SMCF is of Schrödinger type due to the skew-symmetric operator J. Although the SMCF seems to be known by quite many researchers in the field of geometric analysis, little was known about the behaviour of SMCF and many important

problems are left open for quite a long time, especially for the higher dimensional case. In fact, the basic problem of local well-posedness of SMCF for $n \geq 2$ remains unknown.

In this paper, we are mainly concerned on two problems on the SMCF. Namely, the Hasimoto transformation of SMCF and the local existence of SMCF.

Our first goal to generalize the Hasimoto transformation to general SMCFs and try to understand the intrinsic structures of the SMCF from the resulting Schrödinger equation. In particular, one may ask whether the SMCF is integrable in general [18].

By reviewing the classical Hasimoto transformation for vortex filament equation from a geometric point of view, we observe that the classical Hasimoto transformation essentially works by rewriting the evolution equation of the second fundamental form in a suitable gauge. This requires that there exists a global parallel orthonormal frame on the normal bundle, which can always be achieved if the base manifold is one dimensional.

Using this idea, we find that the one dimensional SMCF in a general three manifold $\overline{M}$ is equivalent to the following complex Schrödinger equation

$$-\mathbf{i}\Psi_t = \Psi_{ss} + \frac{1}{2}|\Psi|^2\Psi + W(|\Psi|)\Psi, \tag{1.5}$$

where W is a complex valued function involving the curvature of the ambient space $\overline{M}$.

A consequence which follows from (1.5) and a Strichartz-type estimates is the global existence of a solution to the one dimensional SMCF. This was carried out by H. Gomez in his Ph.D. thesis [11]. The same method is also applied to obtain global existence for one dimensional Schrödinger flow, see [5, 25]. Note that the Strichartz-type esitmate and hence the global existence relies on the fact that equation (1.5) dose not contain first order derivatives of Ψ.

For a general SMCF from an n dimensional Σ with $n \geq 2$, one can not expect to find a global frame of the normal bundle. Thus we investigate the special case when the normal bundle is assumed to be trivial, which includes the important case where $\Sigma = \mathbb{R}^n$. However, even in such a simple case, there dose not exist a global parallel (in the space direction) frame unless the normal bundle is flat. Therefore, we switch to the so-called *temporal gauge* which is parallel in the time direction. Then by a generalized Hasimoto transformation in the temporal gauge, we relate the coefficients of the second fundamental form to a set of complex functions $\Phi = \{\phi_{ij}\}_{1\leq i,j\leq n}$, such that if Σ evolves along the SMCF in the Euclidean space $\mathbb{R}^{n+2}$, then Φ satisfies the Schrödinger system

$$-\mathbf{i}\Phi = \Delta_g\Phi + \Phi * \Phi * \Phi. \tag{1.6}$$

Here $\Delta_g = tr_g\nabla^2$ denotes the covariant Laplacian operator in the normal bundle and $*$ stands for multi-linear maps with coefficients involving the induced metric g.

For a general ambient Riemannian manifold $\overline{M}$, the only difference is that some non-linear, yet zero order terms involving the curvature of $\overline{M}$ would emerge in the non-linear part in equation (1.6). The essential difference of the above Schrödinger system (1.6) from the Schrödinger equation (1.5) in the one dimensional case is that the Laplacian and the non-linear cubic term involves with the induced metric g which is time-dependent. In particular, the covariant Laplacian term $\Delta_g\Phi$ contains the first order derivatives of Φ.

Our second goal, which is the main contribution of this paper, is to show the existence of a local solution to the SMCF of two dimensional surfaces in the Euclidean space $\mathbb{R}^4$. More explicitly, suppose Σ is a two dimensional oriented compact surface and F_0 is a smooth immersion from Σ to

$\mathbb{R}^4$. We consider the initial value problem

$$\begin{cases} \dfrac{\partial F}{\partial t} = J(F)\mathbf{H}(F), \\ F(0,\cdot) = F_0. \end{cases} \tag{1.7}$$

Let $\mathbf{A}_0$ denote the second fundamental form of the immersed surface $F_0(\Sigma)$, we can define a Sobolev-type norm $\|\mathbf{A}_0\|_{H^{2,2}}$ by the induced metric (see (4.20) for details). Our main result is

Theorem 1.1. *Suppose Σ is a two dimensional oriented compact surface. For any smooth immersion $F_0 : \Sigma \to \mathbb{R}^4$, the SMCF (1.7) admits a smooth local solution $F \in C^\infty([0,T] \times \Sigma)$, where the time T only depends on $\|\boldsymbol{A}_0\|_{H^{2,2}}$ and the volume of $F_0(\Sigma)$.*

Since the SMCF is a (weakly) Schrödinger type system, there is no standard existence theory that we can apply, even after we utilize the De Turck trick. Our strategy is to consider a perturbed flow which is (weakly) parabolic, and use the geometric energy method [9, 28], which proved to be a powerful tool in treating geometric Hamiltonian flows. More precisely, for a small real number $\varepsilon > 0$, we will consider the perturbed system

$$\begin{cases} \dfrac{\partial F}{\partial t} = J\mathbf{H} + \varepsilon\mathbf{H}, \\ F(0,\cdot) = F_0. \end{cases} \tag{1.8}$$

The system (1.8) is very similar to the well-studied MCF. A local solution $F_\varepsilon : [0, T_\varepsilon) \times \Sigma \to \mathbb{R}^4$ can be obtained by applying the DeTurk trick and standard parabolic theories. Then we show that F_ε converges to a solution of the original SMCF (1.7) as $\varepsilon \to 0$.

The main difficulty here is to obtain a uniform lower bound for the lifespan T_ε and uniform estimates of F_ε on $[0,T]$. Or equivalently, we need a uniform bound on the second fundamental form $\mathbf{A}_\varepsilon$ along the perturbed SMCF (1.8), at least for a small time $T > 0$ and $\varepsilon > 0$. In fact, once we have a uniform C^0-bound of the second fundamental form $\mathbf{A}_\varepsilon$, the convergence of F_ε and the existence of a solution to the SMCF (1.7) follows from standard arguments, which is analogous to the study of MCF by Huisken [16] and Willmore flow by Kuwert and Schätzle [20].

Our primary observation is the following Sobolev-type embedding theorem for the second fundamental forms of surfaces, which might be of independent interest.

Theorem 1.2. *Given positive numbers B and m, there exists a constant $C(B,m)$, depending only on B and m, such that for any immersed compact surface $\Sigma^2 \subset \mathbb{R}^4$ satisfying*

$$\|\boldsymbol{A}\|_{H^{2,2}} \leq B \text{ and } |\Sigma| \geq m,$$

there holds

$$\|\boldsymbol{A}\|_{C^0} \leq C(B,m).$$

The above theorem holds for two dimensional immersed surfaces in any higher dimensional Euclidean spaces. The key ingredient of the proof is a blow-up analysis technique (cf. [21, 19]) and a compactness theorem of surfaces (cf. [22, 3]). See Section 4.3 below for more details.

The rest of the paper is organized as follows. In Section 2, we show that the SMCF is indeed a Hamiltonian system and prove some basic properties of the SMCF. In Section 3, we first recall the classical Hasimoto transformation for vortex filament equation and then generalized it to one dimensional SMCF in a manifold and higher dimensional SMCF. Next in Section 4, we prove a compactness theorem for surfaces and Theorem 1.2. Then in Section 5, we apply the approximating scheme and study the evolution equations of various geometric quantities under the perturbed SMCF (1.8). Finally, we finish the proof of Theorem 1.1 in Section 5.3.

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2. Preliminaries

2.1. **SMCF as Hamiltonian flow.** A geometric point of view is to regard the SMCF as a Hamiltonian flow in a infinite dimensional symplectic manifold. This is explained in detail in [12] by using the language of non-linear Grassmannians. In fact, given a Riemannian manifold, the space of co-dimension two submanifolds forms an infinite dimensional symplectic manifold. The induced volume of the submanifolds defines a function on the symplectic manifold and its Hamiltonian equation is exactly the SMCF.

More precisely, suppose Σ is an n dimensional manifold and $(\overline{M}, \bar{g})$ is an $(n+2)$ dimensional Riemannian manifold. Let $\mathrm{Imm}(\Sigma, \overline{M})$ denote the space of smooth immersions from Σ into $\overline{M}$ and $\mathcal{I} := \mathrm{Imm}(\Sigma, \overline{M})/\sim$ be its quotient space by identifying immersions with same image. For each immersion $F \in \mathcal{I}$, denote the normal bundle of $F(\Sigma)$ by $\mathcal{N}$ and the pull-back metric on Σ by $g := F^*\bar{g}$. Then the tangent space of $\mathcal{I}$ at F can naturally be identified with the space of smooth sections of the normal bundle $F^*\mathcal{N}$.

Given a volume form $d\bar{\mu}$ on $\overline{M}$, we have a natural symplectic structure, first discovered by Marsden and Weinstein [23] for $n = 1$, on $\mathcal{I}$ defined by

$$\Omega(V, W) = \int_{F(\Sigma)} i_{F_*(V)} i_{F_*(W)} d\bar{\mu}$$

for any $V, W \in T_\Sigma \mathcal{I}$ which are just sections of the normal bundle $F^*\mathcal{N}$. Thus we get an infinite dimensional symplectic manifold $(\mathcal{I}, \Omega)$.

Since there is a Riemannian metric $\bar{g}$ on $\overline{M}$, we have a symplectic structure on $\mathcal{I}$ defined as above using the volume form $d\bar{\mu}$ induced by $\bar{g}$. Similar, we also have an induced metric G on $\mathcal{I}$ given by

$$G(V, W) = \int_{F(\Sigma)} \bar{g}(F_*(V), F_*(W)) d\mu,$$

where $d\mu$ is the volume form of the restricted metric on $F(\Sigma)$. On the other hand, there is a complex structure J on the normal bundle which simply rotate a vector positively by $\pi/2$ in each fiber. Thus we also have a complex structure J on $\mathcal{I}$. More importantly, the symplectic structure Ω, the metric G and the complex structure are compatible, i.e.

$$\Omega(V, W) = G(V, JW).$$

Now the induce volume on Σ defines a function on $\mathcal{I}$ by

$$\mathscr{V}(F) := \int_{F(\Sigma)} d\mu.$$

It is well-known that the mean curvature is the gradient vector field of $\mathscr{V}$ in this setting. Thus the SMCF is just the Hamiltonian flow of the volume function $\mathscr{V}$ in the symplectic manifold $(\mathcal{I}, \Omega)$.

2.2. **The principal symbol of SMCF.** Although it appears at first glance that there is only a tiny difference between the SMCF and the well-known MCF, which comes from the complex structure J, the behaviour of SMCF is totally different from that of MCF. In fact, the SMCF is no longer a (degenerate) parabolic-type equation, but a Schrödinger -type equation since J is skew-symmetric. Here we compute the principal symbol of SMCF in Euclidean space to illustrate the Schrödinger nature of SMCF. The general case is essentially the same.

For an immersion $F : \Sigma \to \mathbb{R}^{n+2}$, we have

$$\Delta_\Sigma F = \mathbf{H},$$

where Δ_Σ is the Laplace operator on Σ of the induced metric. By definition, in local coordinate, the induced metric is given by

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle.$$

Here, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^{n+2}$. Suppose the standard coordinate on $\mathbb{R}^{n+2}$ is given by $\{y^\alpha\}_{\alpha=1}^{n+2}$. Let (g^{ij}) be the inverse matrix of (g_{ij}). Then the Christofell symbol of the induced metric is

$$\begin{aligned} \Gamma_{ij}^k &= \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right\} \\ &= g^{kl} \frac{\partial^2 F^\beta}{\partial x_i \partial x_j} \frac{\partial F^\beta}{\partial x_l}. \end{aligned}$$

Thus we have that

$$\begin{aligned} \Delta_\Sigma F^\alpha &= g^{ij} \left(\frac{\partial^2 F^\alpha}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial F^\alpha}{\partial x_k} \right) \\ &= g^{ij} \left(\frac{\partial^2 F^\alpha}{\partial x_i \partial x_j} - g^{kl} \frac{\partial^2 F^\beta}{\partial x_i \partial x_j} \frac{\partial F^\beta}{\partial x_l} \frac{\partial F^\alpha}{\partial x_k} \right). \qquad (2.1) \end{aligned}$$

Denote $P(F) = J\mathbf{H} = J\Delta_\Sigma F$, then from (2.1), we see that the SMCF (1.1) is a quasilinear system. The linearization operator of P at F is given by

$$D(P)(F)G = Jg^{ij} \left(\frac{\partial^2 G}{\partial x_i \partial x_j} - g^{kl} \left\langle \frac{\partial^2 G}{\partial x_i \partial x_j}, \frac{\partial F}{\partial x_l} \right\rangle \frac{\partial F}{\partial x_k} \right) + \textit{first order terms}.$$

The principal symbol is

$$\begin{aligned} \sigma(D(P))(x,\xi)G &= Jg^{ij} \left(\xi_i \xi_j G - g^{kl} \left\langle G, \frac{\partial F}{\partial x_l} \right\rangle \xi_i \xi_j \frac{\partial F}{\partial x_k} \right) \\ &= |\xi|^2 J \left(G - g^{kl} \left\langle G, \frac{\partial F}{\partial x_l} \right\rangle \frac{\partial F}{\partial x_k} \right) \\ &= |\xi|^2 J (G - G^T) = |\xi|^2 J G^\perp, \qquad (2.2) \end{aligned}$$

where G is an any vector in $\mathbb{R}^{n+2}$, G^T and $G^\perp$ are the tangent part and the normal part of G on Σ. Then we have

$$\langle \sigma(D(P))(x,\xi)G, G \rangle = \left\langle |\xi|^2 JG^\perp, G \right\rangle = |\xi|^2 \left\langle JG^\perp, G^\perp \right\rangle = 0,$$

where we have used the fact that J is an isomorphism from $N\Sigma$ to $N\Sigma$. Thus the principal symbol of P is skew-symmetric. In particular, the SMCF is a (degenerate) Schrödinger type non-linear partial differential equation.

2.3. **Basic properties of SMCF.** In this subsection we show two basic properties of SMCF. Note that these properties hold in arbitrary ambient Riemannian manifold $(\overline{M}, \bar{g})$. For simplicity, we denote the inner product induced by $\bar{g}$ by $\langle \cdot, \cdot \rangle$ and the corresponding Levi-Civita connection on $\overline{M}$ by $\overline{\nabla}$.

Lemma 2.1. *The induced volume form is preserved under the SMCF. In particular, for a compact manifold, the volume is preserved under the SMCF.*

Proof. We prove it pointwise so that we can take normal coordinate around a point $x \in \Sigma$. The induced metric is given by

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle.$$

Since $\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x_j} \right\rangle = 0$, it follows that

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} &= \left\langle \overline{\nabla}_i \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x_j} \right\rangle + \left\langle \frac{\partial F}{\partial x_i}, \overline{\nabla}_j \frac{\partial F}{\partial t} \right\rangle \\ &= -2 \left\langle \frac{\partial F}{\partial t}, \overline{\nabla}_i \frac{\partial F}{\partial x_j} \right\rangle = -2 \left\langle J\mathbf{H}, \mathbf{A}(e_i, e_j) \right\rangle .\end{aligned}$$

Denoting $d\mu$ the induced volume form on Σ, then it is known that

$$\frac{\partial}{\partial t} d\mu = \frac{1}{2} g^{kl} \frac{\partial}{\partial t} g_{kl} d\mu = -\left\langle J\mathbf{H}, \mathbf{H} \right\rangle = 0.$$

This shows that the volume form $d\mu$, and hence the volume $\mathrm{Vol}(\Sigma) := \int_\Sigma d\mu$ is preserved under the SMCF. □

Since the metric on an one dimensional manifold is completely decided by its volume form, we have

Lemma 2.2. *The induced metric is preserved under the 1 dimensional SMCF.*

The next lemma is crucial for the calculations of evolution equations of SMCF.

Lemma 2.3. *The complex structure is parallel w.r.t. the normal connection, i.e.* $\nabla J = 0$.

Proof. It suffices to show that for any locally supported unit normal vector field $\mathbf{V}$ in the normal bundle, we have

$$J \nabla \mathbf{V} = \nabla J \mathbf{V},$$

where ∇ is the induced connection in the normal bundle. Set $\mathbf{W} = J\mathbf{V}$ so that $\mathbf{V} = -J\mathbf{W}$, then $\{\mathbf{V}, \mathbf{W}\}$ forms a local orthonormal frame. Therefore, for any $X \in T(I \times \Sigma)$, we have

$$\begin{aligned}J \nabla_X \mathbf{V} &= J(\overline{\nabla}_X \mathbf{V})^\perp \\ &= J\left(\left\langle \overline{\nabla}_X \mathbf{V}, \mathbf{V} \right\rangle \mathbf{V} + \left\langle \overline{\nabla}_X \mathbf{V}, \mathbf{W} \right\rangle \mathbf{W} \right) \\ &= -\left\langle \overline{\nabla}_X \mathbf{V}, \mathbf{W} \right\rangle \mathbf{V},\end{aligned}$$

and

$$\begin{aligned}\nabla_X (J\mathbf{V}) = \nabla_X \mathbf{W} &= (\overline{\nabla}_X \mathbf{W})^\perp \\ &= \left\langle \overline{\nabla}_X \mathbf{W}, \mathbf{V} \right\rangle \mathbf{V} + \left\langle \overline{\nabla}_X \mathbf{W}, \mathbf{W} \right\rangle \mathbf{W} \\ &= -\left\langle \overline{\nabla}_X \mathbf{V}, \mathbf{W} \right\rangle \mathbf{V}.\end{aligned}$$

This proves the lemma. □

In particular, Lemma 2.3 shows that the complex structure is parallel along the time direction.

3. Hasimoto Transformation

3.1. **Vortex filament equation.** The so-called Hasimoto transformation, which was first discovered by Hasimoto in [15], is a beautiful transformation which relates the vortex filament equation with a cubic Schrödinger equation. It reveals the hidden intrinsic structure of the vortex filament equation, i.e. the 1 dimensional SMCF in $\mathbb{R}^3$. In this subsection, we review the classical Hasimoto transformation from a more geometric view that is readily to be generalized to 1 dimensional SMCF in a three manifold.

Let $\gamma : \mathbb{R}^1 \times [0, T) \to \mathbb{R}^3$ be a curve involving by the vortex filament equation

$$\gamma_t = \gamma_s \times \gamma_{ss} \tag{3.1}$$

where s is the arc-length parameter. Note that by Lemma 2.2, if s is the arc-length parameter at $t = 0$, then s is arc-length parameter for all time $t > 0$. Differentiating (3.1) by s, one easily finds that γ_s satisfies

$$\frac{d}{dt}\gamma_s = \gamma_s \times \gamma_{sss}. \tag{3.2}$$

Denote the normal connection on $N\gamma$ by $\nabla_s := (\overline{\nabla}_s)^{\perp}$ and the complex structure on $N\gamma$ by J. Then (3.2) can be written as

$$\frac{d}{dt}\gamma_s = J\nabla_s\gamma_{ss}.$$

Differentiating the above equation again and using Lemma 2.3, we get

$$\nabla_t\gamma_{ss} = J\nabla_s^2\gamma_{ss} \tag{3.3}$$

where $\nabla_t = (\frac{d}{dt})^{\perp}$ denotes the projection on the normal bundle $N\gamma$.

Equation (3.3) can be regarded as the evolution equation of the mean curvature vector field γ_{ss}, which is a section on the normal bundle $N\gamma$. It can be written more explicitly in a suitable moving frame. Recall that there is a natural Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ along the curve which satisfies

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} \tag{3.4}$$

Here $\{\mathbf{n}, \mathbf{b}\}$ forms an orthnomal frame of the normal bundle $N\gamma$. The normal connection on the normal bundle is given by

$$\nabla_s\mathbf{n} = \tau\mathbf{b},\ \nabla_s\mathbf{b} = -\tau\mathbf{n}.$$

If we write

$$\nabla = d + Ads$$

in this frame, then the connection matrix is given by

$$A = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}.$$

Thus the torsion τ corresponds to the connection of normal connection ∇ in the Frenet frame. Since the bundle $N\gamma$ is trivial (and the curvature vanishes), we can choose a Coulumb gauge of $N\gamma$ globally, such that the torsion vanishes, i.e. the connection form $A \equiv 0$. Indeed, we may let $\phi = \int_0^s \tau ds'$ and rotate $\{\mathbf{n}, \mathbf{b}\}$ by angle $-\phi$ to get

$$\begin{cases} \mathbf{N}_1 & = \cos\phi \cdot \mathbf{n} - \sin\phi \cdot \mathbf{b} \\ \mathbf{N}_2 & = \sin\phi \cdot \mathbf{n} + \cos\phi \cdot \mathbf{b}. \end{cases}$$

Then one readily checks that $\{\mathbf{N}_1, \mathbf{N}_2\}$ forms a parallel frame on the bundle such that

$$\nabla_s\mathbf{N}_1 = \nabla_s\mathbf{N}_2 = 0,$$

Using the parallel frame, we can define the famous Hasimoto transformation which converts a normal vector field $\mathbf{V} = v_1\mathbf{N}_1 + v_2\mathbf{N}_2 \in \Gamma(N\gamma)$ to a complex function by letting

$$\mathrm{HT}(\mathbf{V}) = v_1 + \mathbf{i}v_2.$$

In particular, we may identify the mean curvature vector field with a complex function. More specifically, in this frame we have

$$\gamma_{ss} = k\mathbf{n} = k\cos\phi \cdot N_1 + k\sin\phi \cdot N_2.$$

Then the Hasimoto transformation is given by

$$\Phi(s,t) := \mathrm{HT}(\gamma_{ss}) = ke^{i\int\tau ds} = k\cos\phi + \mathbf{i}\cdot k\sin\phi.$$

If we denote $a^1 = k\cos\phi$, $a^2 = k\sin\phi$, then equation (3.3) becomes

$$a_t^i \mathbf{N}_i + a^i \nabla_t \mathbf{N}_i = a_{ss}^i J\mathbf{N}_i \tag{3.5}$$

Since $J\mathbf{N}_1 = \mathbf{N}_2, J\mathbf{N}_2 = -\mathbf{N}_1$ and $(\nabla_t \mathbf{N}_i, \mathbf{N}_i) = 0$, we get

$$\begin{cases} a_t^1 & = -a_{ss}^2 - a^2(\nabla_t \mathbf{N}_2, \mathbf{N}_1) \\ a_t^2 & = a_{ss}^1 - a^1(\nabla_t \mathbf{N}_1, \mathbf{N}_2). \end{cases}$$

Then we can rewrite (3.5) as a complex equation for Φ and readily checks that Φ satisfies the cubic Schrödinger equation

$$-\mathbf{i}\Phi_t = \Phi_{ss} + \frac{1}{2}|\Phi|^2 \Phi.$$

Remark 3.1. *The advantage of the parallel frame is that the spacial derivative $\nabla_s \boldsymbol{N}_i$ vanishes. However, since these frames are time-dependent, we have to compute $\nabla_t \boldsymbol{N}_i$ as we will do in the next subsection.*

3.2. **SMCF in three manifolds.** Next we apply the Hasimoto transformation to 1 dimensional SMCF in a general three dimensional complete Riemann manifold and derive the corresponding Schrödinger equation.

Let $(\overline{M}, \bar{g})$ be a three manifold with Riemann metric $\bar{g}$ and curvature tensor $\overline{R}$. Let $\gamma(t,s) : [0,T) \times \mathbb{R} \to N$ be a solution to the SMCF. Note that we may choose s to be the arc-length parameter for all time $t \in [0,T)$. Moreover, we denote the connection in $\overline{M}$ by $\overline{\nabla}$ and the normal connection on the normal bundle $N\gamma$ by ∇. Similar to the Euclidean case, there exists a Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ along the curve such that $\mathbf{t} = \gamma_s$ and the frame satisfies equation (3.4). To perform the Hasimoto transform, we denote $\mathbf{T} := \gamma_s = \mathbf{t}$ and fix an orthogonal frame $\{\mathbf{N}_1, \mathbf{N}_2\}$ at $s = 0$ such that

$$J\mathbf{N}_1 = \mathbf{N}_2,\ J\mathbf{N}_2 = -\mathbf{N}_1. \tag{3.6}$$

Again since the normal bundle on γ is trivial, we may extend the frame all over γ by parallel transportation such that

$$\nabla_s \mathbf{N}_i = 0,\ i = 1, 2.$$

Note that the complex structure J is also parallel by Lemma 2.3. Therefore (3.6) holds everywhere and $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2\}$ forms an orthogonal normal frame along γ. For convenience, we adopt the notation

$$J\mathbf{N}_i = \mathbf{N}_{\bar{i}}, i = 1, 2.$$

Suppose $\overline{\nabla}_s \gamma_s = k\mathbf{n} =: a^i \mathbf{N}_i$ under the new frame $\{\mathbf{N}_1, \mathbf{N}_2\}$, where k is the curvature of γ and a^i is a function depending on t and s. Here we assume that $k \neq 0$ everywhere. Then $(a^1)^2 + (a^2)^2 = k^2$ and the SMCF is simply

$$\gamma_t = J\overline{\nabla}_s \gamma_s = a^i J\mathbf{N}_i = a^i \mathbf{N}_{\bar{i}}. \tag{3.7}$$

Since $|\gamma_s|^2 \equiv 1$, we have $\overline{\nabla}_s \gamma_s \perp \gamma_s$ and $\overline{\nabla}_t \gamma_s \perp \gamma_s$. Hence

$$\nabla_s \gamma_s = \overline{\nabla}_s \gamma_s = a^i \mathbf{N}_i. \tag{3.8}$$

Differentiating equation (3.7), we get

$$\begin{aligned} \nabla_t \gamma_s = \nabla_s \gamma_t &= \nabla_s (a^i \mathbf{N}_{\bar{i}}) \\ &= a_s^i \mathbf{N}_{\bar{i}} + a^i \nabla_s \mathbf{N}_{\bar{i}} = a_s^i \mathbf{N}_{\bar{i}}. \end{aligned} \tag{3.9}$$

Differentiating again, we get

$$\nabla_s \nabla_t \gamma_s = \nabla_s (a_s^i \mathbf{N}_{\bar{i}}) = a_{ss}^i \mathbf{N}_{\bar{i}}. \tag{3.10}$$

Since we don't know how to exchange derivatives of the normal connection, we have to turn to the total connection $\overline{\nabla}$ of the ambient space. Note that $\overline{R}(\gamma_s, \gamma_t)\gamma_s \perp \gamma_s$, so that we have

$$
\begin{aligned}
\nabla_s \nabla_t \gamma_s &= \left(\overline{\nabla}_s \overline{\nabla}_t \gamma_s\right)^{\perp} = \left(\overline{\nabla}_t \overline{\nabla}_s \gamma_s + \overline{R}(\gamma_s, \gamma_t)\gamma_s\right)^{\perp} \\
&= \nabla_t \nabla_s \gamma_s + \overline{R}(\gamma_s, \gamma_t)\gamma_s.
\end{aligned}
\tag{3.11}
$$

Combining (3.8), (3.10) and (3.11), we obtain

$$
\begin{aligned}
\nabla_t \nabla_s \gamma_s &= \nabla_t (a^i \mathbf{N}_i) = a^i_t \mathbf{N}_i + a^i \nabla_t \mathbf{N}_i \\
&= a^i_{ss} \mathbf{N}_{\bar{i}} - a^i \overline{R}(\mathbf{T}, \mathbf{N}_{\bar{i}})\mathbf{T}.
\end{aligned}
\tag{3.12}
$$

Next we show how to calculate $\nabla_t \mathbf{N}_i$. First note that since $|\mathbf{N}_i| \equiv 1$, we have $\langle \nabla_t \mathbf{N}_i, \mathbf{N}_i \rangle = 0$. On the other hand, since $\langle \mathbf{N}_1, \mathbf{N}_2 \rangle \equiv 0$, we have $\langle \nabla_t \mathbf{N}_1, \mathbf{N}_2 \rangle = -\langle \mathbf{N}_1, \nabla_t \mathbf{N}_2 \rangle$. Thus we may define a function $b := \langle \nabla_t \mathbf{N}_1, \mathbf{N}_2 \rangle$ such that $\nabla_t \mathbf{N}_i = b \mathbf{N}_{\bar{i}}$ for $i = 1, 2$. Then equation (3.12) becomes

$$
a^i_t \mathbf{N}_i = a^i_{ss} \mathbf{N}_{\bar{i}} - b a^i \mathbf{N}_{\bar{i}} - a^i \overline{R}(\mathbf{T}, \mathbf{N}_{\bar{i}})\mathbf{T}.
\tag{3.13}
$$

To find the function b, we first compute its derivative b_s by

$$
\begin{aligned}
b_s &= \partial_s \left\langle \overline{\nabla}_t \mathbf{N}_1, \mathbf{N}_2 \right\rangle \\
&= \left\langle \overline{\nabla}_s \overline{\nabla}_t \mathbf{N}_1, \mathbf{N}_2 \right\rangle + \left\langle \overline{\nabla}_t \mathbf{N}_1, \overline{\nabla}_s \mathbf{N}_2 \right\rangle \\
&= \left\langle \overline{\nabla}_t \overline{\nabla}_s \mathbf{N}_1, \mathbf{N}_2 \right\rangle + \left\langle \overline{\nabla}_t \mathbf{N}_1, \overline{\nabla}_s \mathbf{N}_2 \right\rangle + \left\langle \overline{R}(\gamma_s, \gamma_t)\mathbf{N}_1, \mathbf{N}_2 \right\rangle \\
&= I + II + III.
\end{aligned}
\tag{3.14}
$$

For the first term, we have

$$
\begin{aligned}
I &= \left\langle \overline{\nabla}_t \overline{\nabla}_s \mathbf{N}_1, \mathbf{N}_2 \right\rangle \\
&= \left\langle \overline{\nabla}_t (\left\langle \overline{\nabla}_s \mathbf{N}_1, \mathbf{T} \right\rangle \mathbf{T}), \mathbf{N}_2 \right\rangle \\
&= \left\langle \left\langle \overline{\nabla}_s \mathbf{N}_1, \mathbf{T} \right\rangle \overline{\nabla}_t \mathbf{T}, \mathbf{N}_2 \right\rangle \\
&= -\left\langle \mathbf{N}_1, \overline{\nabla}_s \mathbf{T} \right\rangle \cdot \left\langle \mathbf{N}_2, \overline{\nabla}_t \mathbf{T} \right\rangle.
\end{aligned}
$$

Similarly for the second term, we have

$$
\begin{aligned}
II &= \left\langle \overline{\nabla}_t \mathbf{N}_1, \overline{\nabla}_s \mathbf{N}_2 \right\rangle \\
&= \left\langle \overline{\nabla}_t \mathbf{N}_1, \left\langle \overline{\nabla}_s \mathbf{N}_2, \mathbf{T} \right\rangle \mathbf{T} \right\rangle \\
&= \left\langle \overline{\nabla}_s \mathbf{N}_2, \mathbf{T} \right\rangle \left\langle \overline{\nabla}_t \mathbf{N}_1, \mathbf{T} \right\rangle \\
&= \left\langle \mathbf{N}_2, \overline{\nabla}_s \mathbf{T} \right\rangle \cdot \left\langle \mathbf{N}_1, \overline{\nabla}_t \mathbf{T} \right\rangle.
\end{aligned}
$$

Recall that we have by (3.9)

$$
\nabla_t \mathbf{T} = \overline{\nabla}_t \mathbf{T} = a^i_s \mathbf{N}_{\bar{i}}
$$

and by(3.8)

$$
\nabla_s \mathbf{T} = \overline{\nabla}_s \mathbf{T} = a^i \mathbf{N}_i.
$$

It follows that

$$
I + II = -\frac{1}{2}\left((a^1)^2 + (a^2)^2\right)_s.
\tag{3.15}
$$

As for the third term in (3.14), since $\gamma_t = a^i \mathbf{N}_{\bar{i}} = k\mathbf{b}$ where k is the curvature and $\mathbf{b}$ is the binormal vector, it is obvious that

$$
III = \left\langle \overline{R}(\mathbf{T}, a^j \mathbf{N}_{\bar{j}})\mathbf{N}_1, \mathbf{N}_2 \right\rangle = k\overline{R}(\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{b}).
\tag{3.16}
$$

Inserting (3.15) and (3.16) into (3.14), we obtain

$$b_s = -\frac{1}{2}\Big((a^1)^2 + (a^2)^2\Big)_s + k\overline{R}(\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{b}).$$

A simple integration yields

$$b = -\frac{1}{2}\Big((a^1)^2 + (a^2)^2\Big) + A(t) + \int_{s_0}^{s} k\overline{R}(\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{b})ds', \tag{3.17}$$

where $A(t)$ is a function only depending on t.

Now apply the Hasimoto transformation and set $\Phi(s,t) = \mathrm{HT}(\nabla_s\gamma_s) = a^1 + \mathbf{i}a^2$. Note that $|\Phi| = k$. From (3.13) and (3.17), it is easy to check that Φ satisfies the equation

$$-\mathbf{i}\Phi_t = \Phi_{ss} + \frac{1}{2}|\Phi|^2\Phi - A(t)\Phi + W(|\Phi|)\Phi, \tag{3.18}$$

where $W(|\Phi|)$ is a complex valued function given by

$$W(|\Phi|) = -\int_{s_0}^{s} |\Phi|\overline{R}(\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{b})ds' + \overline{R}(\mathbf{t}, \mathbf{b}, \mathbf{t}, \mathbf{b}) - \mathbf{i}\overline{R}(\mathbf{t}, \mathbf{b}, \mathbf{t}, \mathbf{n}).$$

The term $-A(t)\Phi$ can be eliminated by setting $\Psi := \Phi \cdot \exp(\mathbf{i}\int_t A(t))$. Indeed, (3.18) is equivalent to

$$-\mathbf{i}\Psi_t = \Psi_{ss} + \frac{1}{2}|\Psi|^2\Psi + W(|\Psi|)\Psi. \tag{3.19}$$

If the sectional curvature of the ambient space is bounded by some constant K, then

$$|W(|\Psi|)| \leq K(\sqrt{2} + \int_{s_0}^{s} kds').$$

3.3. **General case.** From the discussions in previous subsections, we see that the so-called Hasimoto transformation essentially evolves finding a suitable global frame (or gauge) of the normal bundle, which is parallel in the space direction, and then expressing the evolution equation of the mean curvature (or the second fundamental form in higher dimensional case) in the chosen frame. This can always be achieved in the one dimensional case since in this case the normal bundle is always flat. For the higher dimensional SMCF, however, there is generally no analogous Hasimoto transformation that can transform the SMCF to a simple Schrödinger equation. The problem is that there is no global frames on the normal bundle unless additional assumptions are proposed. Generally speaking, we can only choose some special frame locally and the best gauge one can expect is the Coulomb gauge, which dose not help much for the Hasimoto transformation.

Nevertheless, we illustrate here an analogous Hasimoto transformation in high dimensional case here for a special case. Suppose $F : I \times \Sigma^n \to \overline{M}$ is a solution to the SMCF. The additional assumption we need is that the normal bundle $N\Sigma$ is trivial (or parallelizable) such that there exists a global orthonormal frame $\{\nu_{n+1}, \nu_{n+2}\}$ on the normal bundle. For example, if the base manifold Σ is $\mathbb{R}^n$, then this requirement is satisfied.

As we already mentioned, The key ingredient of Hasimoto transformation is to write the evolution equation of the second fundamental form in a properly chosen orthogonal frame. Our observation is that although the frame can not be expected to be parallel in the space direction, we can always choose them to be parallel along the time direction, i.e. the so-called *temporal gauge*.

More explicitly, the pull-back bundle $F^*T\overline{M}$ which is defined on $I \times \Sigma$ splits in an obvious way into the "spacial" subbundle $\mathcal{H}$ and the normal subbundle $\mathcal{N}$. Suppose $\{e_1, \cdots, e_n, \nu_{n+1}, \nu_{n+2}\}$ is a global fame on $F^*T\overline{M}$, such that $\{e_1, \cdots, e_n\}$ is an orthonormal frame of $\mathcal{H}$ and $\{\nu_{n+1}, \nu_{n+2}\}$ is an orthonormal frame of $\mathcal{N}$. We propose the temporal gauge fixing condition

$$\nabla_t^{\mathcal{H}} e_i = \nabla_t^{\mathcal{N}} \nu_\alpha = 0, \quad 1 \leq i, j \leq n, \alpha \in \{n+1, n+2\},$$

where $\nabla^{\mathcal{H}}$ and $\nabla^{\mathcal{N}}$ stands for the induced connections on the $\mathcal{H}$ and $\mathcal{N}$ respectively. The point is that the induced metric g on the tangent bundle and $g^{\perp}$ on the normal bundle are both parallel along the time direction, i.e. $\nabla_t g = \nabla_t g^{\perp} = 0$. Thus $\{e_1, \cdots, e_n, \nu_{n+1}, \nu_{n+2}\}$ remains to be an orthonormal frame for all time $t \in I$. Moreover, the complex structure J on the normal bundle is also parallel along the time direction, i.e. $\nabla_t J = 0$ by Lemma 2.3. In particular, we always have for all time $t \in I$

$$J\nu_{n+1} = \nu_{n+2}, \ J\nu_{n+2} = -\nu_{n+1}.$$

Denote the second fundamental form by $\mathbf{A}$ and its coefficients under the temporal gauge by h^{α}_{ij} where $1 \leq i, j \leq n$ and $\alpha \in \{n+1, n+2\}$. Hence the mean curvature is $\mathbf{H} := tr_g \mathbf{A}$. Moreover, we denote by $\Delta_g = tr_g \nabla^2$ the covariant Laplacian operator in the normal bundle and

$$(\Delta_g \mathbf{A})_{ij} = \Delta_g h^{n+1}_{ij} \nu_{n+1} + \Delta_g h^{n+2}_{ij} \nu_{n+2}.$$

Then we define a set of complex functions $\Phi = \{\phi_{ij}\}$ by the Hasimoto transformation

$$\phi_{ij} := \mathrm{HT}(A_{ij}) = h^{n+1}_{ij} + \mathbf{i} \cdot h^{n+2}_{ij}.$$

Using the parallel properties of the frame, it is not difficult to derive the evolution equation for $\mathbf{A}$ and hence obtain a Schrödinger equation for Φ.

Here we also assume that the ambient space is $\mathbb{R}^{n+2}$ to simplify the structure of the resulting Schrödinger equation. The following result is a direct corollary of Lemma 5.3 below and we omit the proof here. Note that for a general Riemannian manifold $\overline{M}$, the only difference is that some non-linear, yet zero order terms involving the curvature of $\overline{M}$ would immerse in the non-linear part, just as we exhibited in the last subsection for the 1 dimensional case.

Lemma 3.2. *Along the SMCF in $\mathbb{R}^{n+2}$, the second fundamental form under the temporal gauge satisfies*

$$\begin{aligned}\frac{\partial}{\partial t} h^{n+1}_{ij} &= -\Delta_g h^{n+2}_{ij} - h^{n+2}_{im}(h^{\beta}_{mk} h^{\beta}_{kj} - h^{\beta}_{mj} H^{\beta}) - h^{n+2}_{mk}(h^{\beta}_{mk} h^{\beta}_{ij} - h^{\beta}_{ki} h^{\beta}_{mj}) \\ &\quad - h^{\beta}_{ik}(h^{\beta}_{kl} h^{n+2}_{lj} - h^{n+2}_{kl} h^{\beta}_{lj}) - H^{n+2} h^{n+1}_{ik} h^{n+1}_{jk} + H^{n+1} h^{n+1}_{ik} h^{n+2}_{jk}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t} h^{n+2}_{ij} &= \Delta_g h^{n+1}_{ij} - h^{n+1}_{im}(h^{\beta}_{mk} h^{\beta}_{kj} - h^{\beta}_{mj} H^{\beta}) - h^{n+1}_{mk}(h^{\beta}_{mk} h^{\beta}_{ij} - h^{\beta}_{ki} h^{\beta}_{mj}) \\ &\quad - h^{\beta}_{ik}(h^{\beta}_{kl} h^{n+1}_{lj} - h^{n+1}_{kl} h^{\beta}_{lj}) - H^{n+2} h^{n+2}_{ik} h^{n+1}_{jk} + H^{n+1} h^{n+2}_{ik} h^{n+2}_{jk}.\end{aligned}$$

In particular, the complex functions Φ satisfies a Schrödinger system

$$-\boldsymbol{i}\Phi = \Delta_g \Phi + \Phi * \Phi * \Phi, \tag{3.20}$$

where $*$ denotes multi-linear maps with coefficients involving the metric g.

The essential difference between the above Schrödinger equation we obtained for the high dimension case and the standard cubic Schrödinger equation in the 1 dimensional case is that (3.20) is a system of $n \times n$ complex functions $\{\phi_{ij}\}$ instead of a single equation. Moreover, the Laplacian and the non-linear cubic term involves with the induced metric g which is time-dependent. It is worthy to point out that the covariant Laplacian Δ_g generally involves the first order derivatives of Φ. In order to eliminate the first order derivatives, one may hope that the frame $\{\nu_{n+1}, \nu_{n+2}\}$ are parallel such that Δ_g reduces to the usual Laplacian for functions. This requires that the normal bundle is flat. i.e. the normal curvature vanishes everywhere. However, even if we assume that the normal bundle is flat for the initial time $t = 0$, it is not known whether the flatness could be preserved along the SMCF.

4. Estimate of the Second Fundamental Form

4.1. **Estimates for graphs.** In this section, we consider a graph in $(n+m)$-dimensional Euclidean space defined on a n-dimensional domain. The constants emerge in the calculations may depend on the dimension but we will not stress it since n and m are always fixed in the application.

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function, where Ω is a bounded domain on $\mathbb{R}^n$. Then the parametrization of the graph of u, $\Sigma := Graph(u)$, can be represented as

$$F(x_1, \cdots, x_n) = (x_1, \cdots, x_n, u_1(x_1, \cdots, x_n), \cdots, u_m(x_1, \cdots, x_n)).$$

From now on, we will agree on the following index ranges

$$1 \le i, j, k, l \le n, \quad 1 \le \alpha, \beta, \gamma \le m.$$

Denote the the partial derivative of u by $D_i u := D_{x_i} u$. It is easy to see that a basis of the tangent space $T\Sigma$ of Σ can be given by

$$e_i = \frac{\partial F}{\partial x_i} = (0, \cdots, 0, \underbrace{1}_{i}, 0, \cdots, 0, D_i u), \tag{4.1}$$

while a basis of the normal space $N\Sigma$ of Σ can be given by

$$\nu_\alpha = (-Du_\alpha, 0, \cdots, 0, \underbrace{1}_{n+\alpha}, 0, \cdots, 0).$$

By (4.1), the induced metric on $T\Sigma$ is given by

$$g_{ij} = \langle e_i, e_j \rangle = \delta_{ij} + \langle D_i u, D_j u \rangle . \tag{4.2}$$

Here and in the sequel, we will always use $\langle \cdot, \cdot \rangle$ to denote various metrics on $\mathbb{R}^{n+m}$, $\mathbb{R}^n$, $\mathbb{R}^m$ and Σ without any confusion. Similarly, the induced metric on $N\Sigma$ is given by

$$g_{\alpha\beta} = \langle \nu_\alpha, \nu_\beta \rangle = \delta_{\alpha\beta} + \langle Du_\alpha, Du_\beta \rangle . \tag{4.3}$$

We will also use (g^{ij}) and $(g^{\alpha\beta})$ to denote the inverse matrix of (g_{ij}) and $(g_{\alpha\beta})$, respectively. Since

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \left(0, \cdots, 0, \frac{\partial^2 u_1}{\partial x_i \partial x_j}, \cdots, \frac{\partial^2 u_m}{\partial x_i \partial x_j}\right) = (0, D^2_{ij} u),$$

the second fundamental form of Σ is given by

$$\mathbf{A}(e_i, e_j) = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)^{\perp} = g^{\alpha\beta} \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}, \nu_\beta \right\rangle \nu_\alpha = g^{\gamma\beta} \frac{\partial^2 u_\beta}{\partial x_i \partial x_j} \nu_\gamma.$$

Therefore, the component of the second fundamental form is

$$h_{\alpha ij} = \langle \mathbf{A}(e_i, e_j), \nu_\alpha \rangle = \frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}. \tag{4.4}$$

In the following we will use $|\cdot|$ to denote the standard Euclidean metric and $|\cdot|_g$ to denote the induced metric on Σ. From (4.2), we can easily see that the eigenvalues $\{\lambda_i\}_{1 \le i \le n}$ of (g_{ij}) satisfy

$$1 \le \lambda_i \le 1 + |Du|^2. \tag{4.5}$$

Similarly, the eigenvalues $\{\mu_\alpha\}_{1 \le \alpha \le m}$ of $(g_{\alpha\beta})$ satisfy

$$1 \le \mu_\alpha \le 1 + |Du|^2.$$

Therefore, the eigenvalues of (g^{ij}) and $(g^{\alpha\beta})$ can be bounded by

$$\frac{1}{1 + |Du|^2} \le \lambda_i^{-1} \le 1, \quad \frac{1}{1 + |Du|^2} \le \mu_\alpha^{-1} \le 1. \tag{4.6}$$

The next lemma (see also [22], [3], [4]) follows easily from (4.6) and the fact that

$$|\mathbf{A}|_g^2 = g^{ik}g^{jl}g^{\alpha\beta}h_{\alpha ij}h_{\beta kl} = g^{ik}g^{jl}g^{\alpha\beta}\frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}\frac{\partial^2 u_\beta}{\partial x_k \partial x_l}.$$

Lemma 4.1. *Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^2 function. Then we have*

$$|\boldsymbol{A}|_g^2 \le |D^2u|^2 \le (1+|Du|^2)^3|\boldsymbol{A}|_g^2. \tag{4.7}$$

In order to estimate $|\nabla\mathbf{A}|_g^2$ and higher order derivatives, we first need to compute the Christoffel symbols of the tangent bundle $T\Sigma$ and normal bundle $N\Sigma$. We will use $\overline{\nabla}$ to denote the Levi-Civita connection on $\mathbb{R}^{n+m}$, while ∇ the induced connection on Σ. By definition, the Christoffel symbols are given by

$$\nabla_{e_i}e_j = (\overline{\nabla}_{e_i}e_j)^T = \Gamma_{ij}^k e_k, \quad \nabla_{e_i}\nu_\alpha = (\overline{\nabla}_{e_i}\nu_\alpha)^\perp = \Gamma_{i\alpha}^\beta \nu_\beta.$$

Since

$$(\overline{\nabla}_{e_i}e_j)^T = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)^T = g^{kl}\left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}, e_l\right\rangle e_k = g^{kl}\left\langle D_{ij}u, D_l u\right\rangle e_k,$$

we have

$$\Gamma_{ij}^k = g^{kl}\left\langle D_{ij}u, D_l u\right\rangle = g^{kl}\frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}\frac{\partial u_\alpha}{\partial x_l}. \tag{4.8}$$

Similarly, since

$$\begin{aligned}(\overline{\nabla}_{e_i}\nu_\alpha)^\perp &= \left(\frac{\partial}{\partial x_i}\nu_\alpha\right)^\perp = \left(-\frac{\partial^2 u_\alpha}{\partial x_i \partial x_1}, \cdots, -\frac{\partial^2 u_\alpha}{\partial x_i \partial x_n}, 0, \cdots, 0\right)^\perp \\ &= g^{\beta\gamma}\left\langle\left(-\frac{\partial^2 u_\alpha}{\partial x_i \partial x_1}, \cdots, -\frac{\partial^2 u_\alpha}{\partial x_i \partial x_n}, 0, \cdots, 0\right), \nu_\gamma\right\rangle \nu_\beta \\ &= g^{\beta\gamma}\frac{\partial^2 u_\alpha}{\partial x_i \partial x_k}\frac{\partial u_\gamma}{\partial x_k}\nu_\beta,\end{aligned}$$

we have

$$\Gamma_{i\alpha}^\beta = g^{\beta\gamma}\frac{\partial^2 u_\alpha}{\partial x_i \partial x_k}\frac{\partial u_\gamma}{\partial x_k}. \tag{4.9}$$

For simplicity, we may define a vector Γ in some Euclidean space with components given by all Γ_{ij}^k and $\Gamma_{i\alpha}^\beta$. Similarly, we will also treat $\nabla^l\mathbf{A}, D^l u$ and $D^l g^{-1}$ as vectors in (probably different dimensional) Euclidean spaces. In particular, we still use $|\cdot|$ to denote their norms in corresponding Euclidean spaces. Then we can simply write (4.4) as $\mathbf{A} = D^2u$ and rewrite (4.8) and (4.9) as

$$\Gamma = D^2u * Du * g^{-1}, \tag{4.10}$$

where $*$ denotes multiple linear combinations of components of the vectors. It follows from (4.6) that

$$|\Gamma| \le C|D^2u|\cdot|Du|. \tag{4.11}$$

Lemma 4.2. *There exists a constant C such that*

$$|\nabla\boldsymbol{A}|_g \le |D^3u| + C|Du||D^2u|^2,$$

and

$$|D^3u| \le (1+|Du|^2)^2|\nabla\boldsymbol{A}|_g + C|Du||D^2u|^2.$$

Proof. By definition, we have

$$|\nabla \mathbf{A}|_g^2 = h_{\alpha ij,k} h_{\beta pq,l} g^{ip} g^{jq} g^{kl} g^{\alpha\beta}, \tag{4.12}$$

where $h_{\alpha ij,k}$ stands for the k-th covariant derivative of $h_{\alpha ij}$, i.e.

$$h_{\alpha ij,k} = \frac{\partial h_{\alpha ij}}{\partial x_k} + \Gamma^l_{ki} h_{\alpha lj} + \Gamma^l_{kj} h_{\alpha il} + \Gamma^\beta_{k\alpha} h_{\beta ij}.$$

Using our convention, we may simply write

$$\nabla \mathbf{A} = D\mathbf{A} + \Gamma * \mathbf{A} = D^3 u + \Gamma * D^2 u. \tag{4.13}$$

By (4.12) and (4.6), we have

$$|\nabla \mathbf{A}|_g \le |\nabla \mathbf{A}| \le |D^3 u| + C|\Gamma| \cdot |D^2 u|,$$

and

$$|\nabla \mathbf{A}|_g \ge \frac{1}{(1+|Du|^2)^2} |\nabla \mathbf{A}| \ge \frac{1}{(1+|Du|^2)^2} (|D^3 u| - C|\Gamma| \cdot |D^2 u|).$$

Then the lemma follows from (4.11) and Lemma 4.1 easily. □

In order to compute higher order derivatives, we first note that from (4.2),

$$\frac{\partial g_{ij}}{\partial x_k} = \frac{\partial^2 u_\alpha}{\partial x_k \partial x_i} \frac{\partial u_\alpha}{\partial x_j} + \frac{\partial^2 u_\alpha}{\partial x_k \partial x_j} \frac{\partial u_\alpha}{\partial x_i},$$

which implies

$$\frac{\partial g^{ij}}{\partial x_k} = -g^{ip} g^{jq} \left(\frac{\partial^2 u_\alpha}{\partial x_k \partial x_p} \frac{\partial u_\alpha}{\partial x_q} + \frac{\partial^2 u_\alpha}{\partial x_k \partial x_q} \frac{\partial u_\alpha}{\partial x_p} \right).$$

Similarly, from (4.3), we have

$$\frac{\partial g_{\alpha\beta}}{\partial x_k} = \frac{\partial^2 u_\alpha}{\partial x_k \partial x_i} \frac{\partial u_\beta}{\partial x_i} + \frac{\partial^2 u_\beta}{\partial x_k \partial x_i} \frac{\partial u_\alpha}{\partial x_i},$$

which implies

$$\frac{\partial g^{\alpha\beta}}{\partial x_k} = -g^{\alpha\gamma} g^{\beta\delta} \left(\frac{\partial^2 u_\gamma}{\partial x_k \partial x_i} \frac{\partial u_\delta}{\partial x_i} + \frac{\partial^2 u_\delta}{\partial x_k \partial x_i} \frac{\partial u_\gamma}{\partial x_i} \right).$$

Equivalently, we may write

$$Dg^{-1} = D^2 u * Du * g^{-1} * g^{-1}. \tag{4.14}$$

Lemma 4.3. *There exists a constant c_k depending on k, such that*

$$|\nabla^k \boldsymbol{A}|_g \le |D^{k+2} u| + P_k(|Du|) \sum |D^{j_1+1} u| \cdots |D^{j_s+1} u| \tag{4.15}$$

and

$$|D^{k+2} u| \le (1+|Du|^2)^{\frac{k+3}{2}} |\nabla^k \boldsymbol{A}|_g + P_k(|Du|) \sum |D^{j_1+1} u| \cdots |D^{j_s+1} u|, \tag{4.16}$$

where P_k is a polynomial depending on k and the summations are taken over all indices $(j_1, \cdots, j_s)$ satisfying

$$j_1 \ge j_2 \cdots \ge j_s,\ k \ge j_i \ge 1,\ j_1 + j_2 + \cdots j_s = k+1. \tag{4.17}$$

Proof. We prove the lemma by induction. The case $k = 0$ and $k = 1$ has already been proved in Lemma 4.1 and Lemma 4.2 respectively. For $k \ge 2$, first note that

$$|\nabla^k \mathbf{A}|_g^2 = g^{\alpha\beta} g^{p_1 q_1} g^{p_2 q_2} g^{i_1 j_1} \cdots g^{i_k j_k} h_{\alpha p_1 p_2, i_1 \cdots i_k} h_{\beta q_1 q_2, j_1 \cdots j_k}.$$

From (4.6), we see that

$$|\nabla^k \mathbf{A}|_g \le |\nabla^k \mathbf{A}| \le (1+|Du|^2)^{\frac{k+3}{2}} |\nabla^k \mathbf{A}|_g. \tag{4.18}$$

It suffices to estimate $|\nabla^k \mathbf{A}|$. Since

$$\begin{aligned} h_{\alpha p_1 p_2, i_1 \cdots i_k} &= \frac{\partial}{\partial x_{i_k}} h_{\alpha p_1 p_2, i_1 \cdots i_{k-1}} + \Gamma^{\beta}_{i_k \alpha} h_{\beta p_1 p_2, i_1 \cdots i_{k-1}} + \Gamma^{r}_{i_k p_1} h_{\alpha r p_2, i_k \cdots i_{k-1}} \\ &\quad + \Gamma^{r}_{i_k p_2} h_{\alpha p_1 r, i_1 \cdots i_{k-1}} + \sum_{s=1}^{k-1} \Gamma^{r}_{i_k i_s} h_{\alpha p_1 p_2, i_k \cdots i_{s-1} r i_{s+1} \cdots i_{k-1}}, \end{aligned}$$

we get

$$\nabla^k \mathbf{A} = D\nabla^{k-1}\mathbf{A} + \Gamma * \nabla^{k-1}\mathbf{A}. \tag{4.19}$$

Using (4.13) and (4.19), we can verify by induction that

$$\nabla^k \mathbf{A} = D^{k+2}u + \tilde{P}_k(g^{-1}, Du, D^2u, \cdots, D^{k+1}u).$$

where $\tilde{P}_k$ are multiple linear form given by

$$\tilde{P}_k = \underbrace{g^{-1} * \cdots g^{-1}}_{k} * \underbrace{Du * \cdots Du}_{k} * \sum D^{j_1+1}u * \cdots * D^{j_s+1}u$$

with the summation taken over indices satisfying (4.17). Therefore, there exists a polynomial P_k depending only on k, such that

$$|\tilde{P}_k| \le P_k(|Du|) \sum |D^{j_1+1}u| \cdots |D^{j_s+1}u|.$$

Now (4.15) and (4.16) follows from (4.18). $\square$

Given a graph Σ represented by a function $u : D_r \to \mathbb{R}^m$, where $D_r \subset \mathbb{R}^n$ is a disk centered at the origin with radius $r > 0$. By (4.4), we may identify $\mathbf{A} = D^2u$. For any non-negative integer k and positive number p, there is a usual Sobolev norm of the Hession D^2u given by

$$\|D^2u\|_{W^{k,p}} = \left(\int_{D_r} \sum_{l=0}^{k} |D^l D^2 u|_g^p dx \right)^{\frac{1}{p}}.$$

On the other hand, we can define a Sobolev-type norm of $\mathbf{A}$ by

$$\|\mathbf{A}\|_{H^{k,p}} = \left(\int_{\Sigma} \sum_{l=0}^{k} |\nabla^l \mathbf{A}|_g^p d\mu \right)^{\frac{1}{p}}. \tag{4.20}$$

In particular, we have $\|\mathbf{A}\|_{L^p} := \|\mathbf{A}\|_{H^{0,p}}$.

The next lemma shows that if $|Du|$ is bounded, then the Sobolev norms of $\mathbf{A}$ can be bounded by the usual Sobolev norms of the Hessian D^2u. The proof of the lemma follows closely that of Lemma 2.2 in [9].

Lemma 4.4. *Let $r > 0, \alpha > 0, \beta > 0$ be positive numbers. Suppose Σ is a smooth graph represented by $u : D_r \subset \mathbb{R}^2 \to \mathbb{R}^m$ with $|Du| \le \alpha$, then for any $k \ge 0$,*

$$\|\boldsymbol{A}\|_{H^{k,2}} \le C \sum_{s=1}^{k+1} \|D^2u\|^s_{W^{k,2}}, \tag{4.21}$$

where the constant C depends on r, α and k.

Proof. For convenience, set $\sigma := D^2u$. Then the inequality (4.15) can be written as

$$|\nabla^k \mathbf{A}|_g \le |D^k\sigma| + P_k(|Du|)\sum |D^{j_1-1}\sigma|\cdots|D^{j_s-1}\sigma|,$$

where the summation is taken over all indices satisfying (4.17). Since $|Du| \le \alpha$, we may integrate to get

$$\|\nabla^k \mathbf{A}\|_{L^2} \le \|D^k\sigma\|_{L^2} + C\sum \left\||D^{j_1-1}\sigma|\cdots|D^{j_s-1}\sigma|\right\|_{L^2}, \tag{4.22}$$

where C depends on k and α. For the second term in the last inequality, we may apply Hölder's inequality to get

$$\left\||D^{j_1-1}\sigma|\cdots|D^{j_s-1}\sigma|\right\|_{L^2} \le \|D^{j_1-1}\sigma\|_{L^{q_1}}\cdots\|D^{j_s-1}\sigma\|_{L^{q_s}}, \tag{4.23}$$

where the numbers $q_1,\cdots q_s$ satisfies

$$\frac{1}{q_1}+\cdots+\frac{1}{q_s} = \frac{1}{2}.$$

We claim that the numbers q_i can be chosen such that there exists $\frac{j_i-1}{k} \le a_i \le 1$ satisfying the equality

$$\frac{1}{q_i} = \frac{j_i-1}{2} + a_i(\frac{1}{2}-\frac{k}{2}) + (1-a_i)\frac{1}{2}. \tag{4.24}$$

If the claim is true, then we can apply the Gagliardo-Nirenberg interpolation inequality(see for example [10], page 27, Theorem 10.1) to σ to get

$$\|D^{j_i-1}\sigma\|_{L^{q_i}} \le C\|\sigma\|^{a_i}_{W^{k,2}}\|\sigma\|^{1-a_i}_{L^2} \le C\|\sigma\|_{W^{k,2}} \tag{4.25}$$

where the constant C depends on r, k, j_i and q_i. Combining (4.22), (4.23) and (4.25), we see that (4.21) follows easily.

The proof of the above claim is a simple calculation and we refer to [9], page 1452. □

4.2. **Compactness results for surfaces.** Next, we restrict ourselves to the case of codimension two surfaces in $\mathbb{R}^4$, i.e. the case $n = m = 2$. We first show that in this case, the opposite of Lemma 4.4 is also correct. Namely, if in addition $|D^2u|$ is bounded, then the Sobolev norms of $\mathbf{A}$ and D^2u are equivalent.

Lemma 4.5. *Let $r > 0, \alpha > 0, \beta > 0$ be positive numbers. Suppose Σ is a smooth graph represented by $u : D_r \subset \mathbb{R}^2 \to \mathbb{R}^m$ with $|Du| \le \alpha$ and $|D^2u| \le \beta$, then for any $k \ge 0$,*

$$\|D^2u\|_{W^{k,2}} \le C\sum_{s=1}^{k}\|\boldsymbol{A}\|^s_{H^{k,2}}, \tag{4.26}$$

where the constant C depends on r, α, β and k.

Proof. We will prove (4.26) by induction. As before, we set $\sigma := D^2u$.

The case $k = 0$ follows directly from (4.5) and (4.7), i.e.,

$$\|\sigma\|_{L^2} \le C_0\|\mathbf{A}\|_{L^2}. \tag{4.27}$$

By our assumption and Lemma 4.2, we have

$$|D\sigma| \le C(|\nabla\mathbf{A}|_g + |\mathbf{A}|_g).$$

Thus the case $k = 1$ also holds true.

Now assume by induction that (4.26) holds for any $k \ge 1$. To prove the lemma, it suffices to estimate $\|D^{k+1}\sigma\|_{L^2}$ in terms of $\|\mathbf{A}\|_{H^{k+1,2}}$.

Recall that by (4.16), we have

$$\|D^{k+1}\sigma\|_{L^2} \le C\|\nabla^{k+1}\mathbf{A}\|_{L^2} + C\sum \left\| |D^{j_1-1}\sigma|\cdots|D^{j_s-1}\sigma| \right\|_{L^2}, \tag{4.28}$$

where the summations are taken over all indices $(j_1, \cdots, j_s)$ satisfying

$$j_1 \ge j_2 \cdots \ge j_s,\ k+1 \ge j_i \ge 1,\ j_1 + j_2 + \cdots j_s = k+2. \tag{4.29}$$

To estimate the second term in the right hand side of (4.28), we consider two cases:

Case 1: $j_1 = k+1$: In this case, by (4.29), it obvious that $s = 2$ and $j_2 = 1$. Then the term is simply bounded by

$$\left\| |D^k\sigma| \cdot |\sigma| \right\|_{L^2} \le \|\sigma\|_{L^\infty}\|D^k\sigma\|_{L^2} \le \beta C \sum_{s=1}^{\gamma_k} \|\mathbf{A}\|^s_{H^{k,2}},$$

where the last inequality used the induction assumption.

Case 2: $j_1 < k+1$: In this case, by (4.29), we have $1 \le j_i \le k$ for any $1 \le i \le s$. Applying Hölder's inequality, we get

$$\left\| |D^{j_1-1}\sigma|\cdots|D^{j_s-1}\sigma| \right\|_{L^2} \le \|D^{j_1-1}\sigma\|_{L^{q_1}}\cdots\|D^{j_s-1}\sigma\|_{L^{q_s}}, \tag{4.30}$$

where the numbers $q_1, \cdots q_s \in [2, \infty]$ satisfies

$$\frac{1}{q_1} + \cdots + \frac{1}{q_s} = \frac{1}{2}. \tag{4.31}$$

Then we can find number a_i decided by the following equality

$$\frac{1}{q_i} = \frac{j_i - 1}{2} + a_i\left(\frac{1}{2} - \frac{k}{2}\right) + (1 - a_i)\frac{1}{2} = \frac{j_i - ka_i}{2}. \tag{4.32}$$

Since $2 \le q_i \le +\infty$, one can easily verify that $\frac{j_i-1}{k} \le a_i \le 1$. Therefore, we may apply the Gagliardo-Nirenberg interpolation inequality(see for example [10], page 27, Theorem 10.1) to σ to get

$$\|D^{j_i-1}\sigma\|_{L^{q_i}} \le C\|\sigma\|^{a_i}_{W^{k,2}}\|\sigma\|^{1-a_i}_{L^2} \tag{4.33}$$

where the constant C depends on r, k, j_i and q_i. Putting (4.33) into (4.30) yields

$$\left\| |D^{j_1-1}\sigma|\cdots|D^{j_s-1}\sigma| \right\|_{L^2} \le C\|\sigma\|^{\sum_i a_i}_{W^{k,2}}\|\sigma\|^{\sum_i(1-a_i)}_{L^2}.$$

By (4.31) and (4.32), it is easy to see that

$$\sum_i a_i = 1 + \frac{1}{k}, \quad \sum_i(1 - a_i) = s - 1 - \frac{1}{k}.$$

Since $2 \le s \le k+2$, it follows

$$\sum \left\| |D^{j_1-1}\sigma|\cdots|D^{j_s-1}\sigma| \right\|_{L^2} \le C\|\sigma\|^{1+\frac{1}{k}}_{W^{k,2}}\|\sigma\|^{s-1-\frac{1}{k}}_{L^2}. \tag{4.34}$$

Combining (4.28) and (4.34), and noting that

$$\|\sigma\|_{L^2} \le \|\sigma\|_{L^\infty}|D_r|^{\frac{1}{2}} \le \beta\sqrt{\pi}r,$$

we get

$$\|D^{k+1}\sigma\|_{L^2} \le C\|\nabla^{k+1}\mathbf{A}\|_{L^2} + C\|\sigma\|^{1+\frac{1}{k}}_{W^{k,2}}.$$

Using the induction assumption, we conclude that (4.26) holds for $k+1$ and the lemma follows. □

Then following Langer [22], we can prove the following compactness theorem.

Theorem 4.6. *Given a compact two dimensional surface Σ, an integer $k \geq 1$ and constants $\beta, \mathcal{A}, \mathcal{V} > 0$, let $\mathcal{M}$ be the set of immersions $F : \Sigma \to \mathbb{R}^4$ satisfying $\|\boldsymbol{A}\|_{L^\infty} \leq \beta$, $\|\boldsymbol{A}\|_{H^{k,2}} \leq \mathcal{A}$, $vol(\Sigma) \leq \mathcal{V}$ and $0 \in F(\Sigma)$. Then for any sequence F_i in $\mathcal{M}$, there exists a sequence of diffeomorphisms ϕ_i on Σ, such that $F_i \circ \phi_i$ sub-converges in $W^{k+2,2}$ weakly and $C^{k,\alpha}$ strongly to an immersion $F_\infty \in \mathcal{M}$, where $0 < \alpha < 1$.*

There is also a localized version of the above theorem which is useful in blow-up analysis.

Theorem 4.7. *Given a compact two dimensional surface Σ, an integer $k \geq 1$ and constants $\beta, \mathcal{A}, \mathcal{V}(R) > 0$ where $\mathcal{V}(R)$ depends on R, let $F_i : \Sigma \to \mathbb{R}^4$ be a sequence of immersions satisfying $\|\boldsymbol{A}(F_i)\|_{L^\infty} \leq \beta$, $\|\boldsymbol{A}(F_i)\|_{H^{k,2}} \leq \mathcal{A}$, $0 \in F_i(\Sigma)$ and*

$$vol(\Sigma_i(R)) \leq C(R)$$

where $\Sigma_i(R) = \Sigma_i \cap B(R)$ is the portion of the immersed surface $\Sigma_i := F_i(\Sigma)$ bounded in the Euclidean ball of radius R. Then there exists a surface $\tilde{\Sigma}$ without boundary, an immersion $F_\infty : \tilde{\Sigma} \to \mathbb{R}^4$ and a sequence of diffeomorphisms ϕ_i, such that $F_i \circ \phi_i$ sub-converges to F on any compact subset of $\tilde{\Sigma}$ in $W^{k+2,2}$ weakly and $C^{k,\alpha}$ strongly, where $0 < \alpha < 1$. Here $\phi_i : U_i \to F_i^{-1}(\Sigma_i(R))$ are defined on open sets $U_i \subset \tilde{\Sigma}$ where $U_i \subset\subset U_{i+1}$ and $\tilde{\Sigma} = \cup_{i=1}^\infty U_i$.

For simplicity, we say that F_i converges to F_∞ weakly in $W^{k+2,2}$-topology and strongly in $C^{k,\alpha}$-topology to F_∞ in Theorem 4.6, and F_i converges locally to F_∞ in the same topology in Theorem 4.7, respectively.

To prove the above theorems, one first view the surface as a system of graphs defined locally on the tangent spaces. Since the second fundamental forms has uniform upper bounds, it follows from a theorem of Langer [22] that there exists a uniform pair of number $r > 0$ and $\alpha > 0$, such that each Σ_i is a (r, α)-immersion. Namely, for any point $y \in \Sigma_i$, there is a neighborhood of y which can be represented by a graph $u : D_r \to \mathbb{R}^2$ such that $|Du| < \alpha$. On each disk D_r, we can apply Lemma 4.5 to find that u has uniformly bounded $W^{k+2,2}$-norms in terms of $\|\mathbf{A}\|_{H^{k,2}}$. Therefore, the graphs converge weakly in $W^{k+2,2}(D_r)$ and strongly in $C^{k,\alpha}(D_r)$. Next, we can patch the system of graphs together to construct the limit immersed surface. Here we omit the details and refer the readers to Breuning's paper [3] for a proof.

4.3. **Uniform estimate for second fundamental form.** Recall that in [9], Ding and Wang generalized the classical Gagliardo-Nirenberg interpolation inequality to sections of vector bundles. In particular, if we regard the second fundamental form $\mathbf{A}$ as a section of the bundle $T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma$, then it follows from [9] that an interpolation inequality holds for $\mathbf{A}$. However, if the metric of the underlying manifold is varying, which is the case of SMCF, the Sobolev constant will vary.

Here we use blow up techniques to establish a uniform embedding theorem for $\mathbf{A}$ under suitable assumptions, which will play a crucial role in the proof of our main theorem 1.1. The blow up techniques applied here is analogous to the one used in the study of Willmore surfaces, see for example [21] [19]. From now on, we simply denote the induced volume of Σ by $|\Sigma| := \mathrm{Vol}(\Sigma)$.

Theorem 4.8. *Given positive numbers B and m, there exists a constant $C(B, m)$ such that for any immersed compact surface $\Sigma^2 \subset \mathbb{R}^4$ satisfying*

$$\|\boldsymbol{A}\|_{H^{2,2}} \leq B \text{ and } |\Sigma| \geq m,$$

there holds

$$\|\boldsymbol{A}\|_{C^0} \leq C(B, m).$$

Proof. We argue by contradiction and apply a blowing-up technique following Section 4 of [21]. Suppose the theorem is false. Then there exists a sequence of compact surfaces $\Sigma_k \subset \mathbb{R}^4$ with $\|\mathbf{A}_k\|_{W^{2,2}} \leq B$ and $|\Sigma_k| \geq m$ such that $\lim_{k\to\infty} \|\mathbf{A}_k\|_{C^0} = +\infty$. We are going to show that this is impossible by using blow-up analysis.

Since Σ_k is compact, $\|\mathbf{A}_k\|_{C^0}$ is attained at some point $y_k \in \Sigma_k$ such that

$$|\mathbf{A}_k(y_k)| = \max_{y\in\Sigma_k} |\mathbf{A}_k(y)|.$$

Denote $r_k := 1/|\mathbf{A}_k(y_k)|$. Then $\lim_{k\to\infty} r_k = 0$. Thus we can define a sequence of rescaled surfaces $\Sigma'_k = \frac{\Sigma_k - y_k}{r_k}$. Denote the corresponding second fundamental form by $\mathbf{A}'_k$. Let g_k and g'_k be the induced metric on Σ_k and Σ'_k respectively. By rescaling properties, we have $g_k = r_k^2 g'_k$, $\mathbf{A}_k = r_k^2 \mathbf{A}'_k$ and $\nabla_{g_k}\mathbf{A}_k = r_k^3 \nabla_{g'_k}\mathbf{A}'_k$. It follows easily that

$$|\mathbf{A}'_k|_{g'_k} = r_k|\mathbf{A}_k|_{g_k} \le r_k|\mathbf{A}_k(y_k)|_{g_k} = 1. \tag{4.35}$$

and

$$|\Sigma'_k|_{g'_k} = r_k^{-2}|\Sigma_k|_{g_k} \tag{4.36}$$

Moreover, we have

$$\|\mathbf{A}'_k\|_{L^2,g'_k} = \|\mathbf{A}_k\|_{L^2,g_k} \tag{4.37}$$

and

$$\|\nabla_{g'_k}\mathbf{A}'_k\|_{L^2,g'_k} = r_k\|\nabla_{g_k}\mathbf{A}_k\|_{L^2,g_k}. \tag{4.38}$$

One can verify that the sequence of rescaled surfaces Σ'_k satisfies all the requirements of Theorem 4.7. (The local volume bound follows from Simon's inequality (1.3) in [27].) Then it follows that there exists a subsequence of the surfaces, which we still denote by Σ'_k, such that Σ'_k converges locally to a complete surface Σ_0 weakly in $W^{4,2}$ and strongly in $C^{2,\alpha}$.

Now let $\mathbf{A}_0$ and g_0 denote the second fundamental form and induced metric of the limit surface Σ_0 respectively. It follows from (4.35) and (4.36) that

$$\|\mathbf{A}_0\|_{C^0,g_0} = \lim_{k\to\infty} r_k|\mathbf{A}_k(y_k)|_{g_k} = 1. \tag{4.39}$$

and

$$|\Sigma_0|_{g_0} = \lim_{k\to\infty} r_k^{-2}|\Sigma_k|_{g_k} \ge r_k^{-2} m = +\infty. \tag{4.40}$$

Moreover, by (4.37) and (4.38), we have

$$\|\mathbf{A}_0\|_{L^2,g_0} = \lim_{k\to\infty} \|\mathbf{A}_k\|_{L^2,g_k} \le B \tag{4.41}$$

and

$$\|\nabla_{g_0}\mathbf{A}_0\|_{L^2,g_0} = \lim_{k\to\infty} r_k\|\nabla_{g_k}\mathbf{A}_k\|_{L^2,g_k} \le \lim_{k\to\infty} r_k B = 0. \tag{4.42}$$

Note that by Kato's inequality, we have

$$|\nabla_{g_0}|\mathbf{A}_0|_{g_0}| \le |\nabla_{g_0}\mathbf{A}_0|_{g_0},$$

which together with (4.42) yields

$$\int_{\Sigma_0} |\nabla_{g_0}|\mathbf{A}_0|_{g_0}|^2 d\mu_{g_0} \le \|\nabla_{g_0}\mathbf{A}_0\|^2_{L^2,g_0} = 0.$$

Thus we find $\nabla_{g_0}|\mathbf{A}_0|_{g_0} = 0$ a.e. on Σ_0, which implies that $|\mathbf{A}_0|_{g_0}$ is constant on Σ_0. It follows from (4.39) that $|\mathbf{A}_0|_{g_0} \equiv 1$. However, this together with (4.40) would imply

$$\|\mathbf{A}_0\|_{L^2,g_0} = |\Sigma_0|_{g_0} = +\infty,$$

which is impossible in view of (4.41). $\square$

Remark 4.9. *Obviously, using Theorem 4.8, we can replace the requirement of upper bound on $\|\boldsymbol{A}\|_{L^\infty}$ with a lower bound on the volume of the surfaces, then the compactness results in Theorem 4.6 and Theorem 4.7 still holds.*

5. Short Time Existence of SMCF

5.1. **The perturbed flow.** For the moment we consider the general n dimensional SMCF in $\mathbb{R}^{n+2}$. To obtain a local solution to the SMCF (1.7), we will consider the perturbed SMCF (1.8)

$$\begin{cases} \dfrac{\partial F}{\partial t} = J\mathbf{H} + \varepsilon \mathbf{H}, \\ F(0,\cdot) = F_0, \end{cases} \tag{5.1}$$

where $\varepsilon > 0$ is a positive number. The idea is to solve the perturbed SMCF (1.8) and approach the original SMCF (1.7) by letting ε go to zero.

Similar to the argument in Section 2.2, it is easy to check that the system (5.1) is a degenerate parabolic system. The degeneracy of the equation is caused by the diffeomorphism group of the underlying manifold, just as in the case of mean curvature flow(MCF). It is well-know that by applying the DeTurk's trick, one can prove the short time existence of a solution to the MCF. Here we follow the same trick to show the existence of a local solution of the perturbed SMCF (1.8).

Lemma 5.1. *For each $\varepsilon > 0$, the Cauchy problem (1.8) admits a unique smooth solution on the time interval $[0, T_\varepsilon)$ for some $T_\varepsilon > 0$.*

Proof. Similar to the arguments in Section 2.2, especially following (2.2), we see that, if we set $P_\varepsilon(F) = J\mathbf{H} + \varepsilon\mathbf{H} = J\Delta_\Sigma F + \varepsilon \Delta_\Sigma F$, then the principal symbol of P_ε is:

$$\sigma(D(P_\varepsilon))(x,\xi)G = |\xi|^2(JG^\perp + \varepsilon G^\perp).$$

Therefore, we have:

$$\langle \sigma(D(P_\varepsilon))(x,\xi)G, G\rangle = \left\langle |\xi|^2(JG^\perp + \varepsilon G^\perp), G\right\rangle = |\xi|^2 \left\langle JG^\perp + \varepsilon G^\perp, G^\perp\right\rangle = \varepsilon|\xi|^2|G^\perp|^2.$$

This shows that for each fixed $\varepsilon > 0$, the operator P_ε is an elliptic operator module tangential diffeomorphisms of the submanifold Σ. Applying De Turck's trick and standard parabolic theory (see for example chapter 15 of [29]), we know that for each smooth initial data, the Cauchy problem (5.1) admits a unique smooth local solution. □

5.2. **Evolution equations.** In this subsection, we will calculate the evolution equation of various geometric quantities for the perturbed SMCF (1.8). Since calculations are standard as in the case of MCF, we only provide sketches here. Note that Lemma 2.3 is crucial in the calculations since we can always commute the normal connection ∇ and the complex structure J.

Choose a local field of orthonormal frames $e_1, \cdots, e_n, \nu_{n+1}, \nu_{n+2}$ of $\mathbf{R}^{n+1}$ along Σ_t such that $e_1, \cdots, e_n$ are tangent vectors of Σ_s and ν_{n+1}, ν_{n+2} are in the normal bundle over Σ_t. From now on, we will agree on the following index ranges:

$$1 \le i,j,k,l \le n, \quad n+1 \le \alpha,\beta,\gamma \le n+2, \quad 1 \le A,B,C \le n+2.$$

We also assume that the frame is chosen so that $J\nu_{n+1} = \nu_{n+2}$, $J\nu_{n+2} = -\nu_{n+1}$. It is not hard to check that, we can always choose the normal frame $\{\nu_{n+1}, \nu_{n+2}\}$ so that $\left\langle \frac{d}{dt}\nu_\alpha, \nu_\beta\right\rangle = 0$, i.e., $\nabla_t \nu_\alpha = 0$.

Using the frame, we denote

$$\mathbf{V} = J\mathbf{H} + \varepsilon\mathbf{H} = V^\alpha e_\alpha,$$

so that

$$V^{n+1} = \varepsilon H^{n+1} - H^{n+2}, \quad V^{n+2} = \varepsilon H^{n+2} + H^{n+1}. \tag{5.2}$$

Lemma 5.2. *Denote $g = (g_{ij})$ the induced metric on Σ. Then along the perturbed SMCF (1.8), we have*

$$\frac{\partial}{\partial t} g_{ij} = -2 \langle J\boldsymbol{H}, \boldsymbol{A}(e_i, e_j) \rangle - 2\varepsilon \langle \boldsymbol{H}, \boldsymbol{A}(e_i, e_j) \rangle . \tag{5.3}$$

As a consequence, we have

$$\frac{\partial}{\partial t} d\mu = -\varepsilon |\boldsymbol{H}|^2 d\mu, \tag{5.4}$$

where $d\mu$ is the induced volume form on Σ.

Proof. We prove it pointwise so that we can take normal coordinate around a point $x \in \Sigma$. The induced metric is given by

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle .$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= \left\langle \frac{\partial}{\partial x_i} \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x_j} \right\rangle + \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j} \frac{\partial F}{\partial t} \right\rangle \\ &= -2 \left\langle \frac{\partial F}{\partial t}, \mathbf{A}(e_i, e_j) \right\rangle = -2 \langle J\mathbf{H}, \mathbf{A}(e_i, e_j) \rangle - 2\varepsilon \langle \mathbf{H}, \mathbf{A}(e_i, e_j) \rangle . \end{aligned}$$

Denoting $d\mu$ the induced volume form on Σ, then it is known that

$$\frac{\partial}{\partial t} d\mu = \frac{1}{2} g^{kl} \frac{\partial}{\partial t} g_{kl} d\mu = (-\langle J\mathbf{H}, \mathbf{H} \rangle - \varepsilon \langle \mathbf{H}, \mathbf{H} \rangle) d\mu = -\varepsilon |\mathbf{H}|^2 d\mu.$$

□

Lemma 5.3. *Along the perturbed SMCF (1.8), the second fundamental form satisfies*

$$\begin{aligned} \frac{\partial}{\partial t} h^{n+1}_{ij} &= \varepsilon \Delta h^{n+1}_{ij} - \Delta h^{n+2}_{ij} + \varepsilon h^{n+1}_{im} (h^{\beta}_{mk} h^{\beta}_{kj} - h^{\beta}_{mj} H^{\beta}) + \varepsilon h^{n+1}_{mk} (h^{\beta}_{mk} h^{\beta}_{ij} - h^{\beta}_{ki} h^{\beta}_{mj}) \\ &\quad + \varepsilon h^{\beta}_{ik} (h^{\beta}_{kl} h^{n+1}_{lj} - h^{n+1}_{kl} h^{\beta}_{lj}) - h^{n+2}_{im} (h^{\beta}_{mk} h^{\beta}_{kj} - h^{\beta}_{mj} H^{\beta}) - h^{n+2}_{mk} (h^{\beta}_{mk} h^{\beta}_{ij} - h^{\beta}_{ki} h^{\beta}_{mj}) \\ &\quad - h^{\beta}_{ik} (h^{\beta}_{kl} h^{n+2}_{lj} - h^{n+2}_{kl} h^{\beta}_{lj}) - V^{\beta} h^{n+1}_{ik} h^{\beta}_{jk}, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} h^{n+2}_{ij} &= \varepsilon \Delta h^{n+2}_{ij} + \Delta h^{n+1}_{ij} + \varepsilon h^{n+2}_{im} (h^{\beta}_{mk} h^{\beta}_{kj} - h^{\beta}_{mj} H^{\beta}) + \varepsilon h^{n+2}_{mk} (h^{\beta}_{mk} h^{\beta}_{ij} - h^{\beta}_{ki} h^{\beta}_{mj}) \\ &\quad + \varepsilon h^{\beta}_{ik} (h^{\beta}_{kl} h^{n+2}_{lj} - h^{n+2}_{kl} h^{\beta}_{lj}) - h^{n+1}_{im} (h^{\beta}_{mk} h^{\beta}_{kj} - h^{\beta}_{mj} H^{\beta}) - h^{n+1}_{mk} (h^{\beta}_{mk} h^{\beta}_{ij} - h^{\beta}_{ki} h^{\beta}_{mj}) \\ &\quad - h^{\beta}_{ik} (h^{\beta}_{kl} h^{n+1}_{lj} - h^{n+1}_{kl} h^{\beta}_{lj}) - V^{\beta} h^{n+2}_{ik} h^{\beta}_{jk}. \end{aligned} \tag{5.6}$$

In particular, if we denote $(J\boldsymbol{A})_{ij} = J(h^{n+1}_{ij} \nu_{n+1} + h^{n+2}_{ij} \nu_{n+2}) = -h^{n+2}_{ij} \nu_{n+1} + h^{n+1}_{ij} \nu_{n+2}$, then (5.5) and (5.6) can be written as

$$\frac{\partial}{\partial t} \boldsymbol{A} = \varepsilon \Delta \boldsymbol{A} + J \Delta \boldsymbol{A} + \boldsymbol{A} * \boldsymbol{A} * \boldsymbol{A}. \tag{5.7}$$

Proof. From Lemma 8.3 of [1], we see that

$$\frac{\partial}{\partial t} h^{n+1}_{ij} = -V^{n+1}_{,ji} + V^{\beta} h^{n+1}_{ik} h^{\beta}_{jk} + h^{\beta}_{ij} \left\langle e_{\beta}, \overline{\nabla}_{\mathbf{V}} e_{n+1} \right\rangle ,$$

and

$$\frac{\partial}{\partial t} h^{n+2}_{ij} = -V^{n+2}_{,ji} + V^{\beta} h^{n+2}_{ik} h^{\beta}_{jk} + h^{\beta}_{ij} \left\langle e_{\beta}, \overline{\nabla}_{\mathbf{V}} e_{n+2} \right\rangle .$$

By our choice of the normal frame, the last terms will disappear in the above two identities. Remember the following commutation formula (see for example Page 332 of [32])

$$\Delta h^\alpha_{ij} = H^\alpha_{,ij} + h^\alpha_{im}(h^\gamma_{mj}H^\gamma - h^\gamma_{mk}h^\gamma_{kj}) + h^\alpha_{mk}(h^\gamma_{mj}h^\gamma_{ik} - h^\gamma_{mk}h^\gamma_{ij}) + h^\beta_{ik}(h^\beta_{lj}h^\alpha_{lk} - h^\beta_{lk}h^\alpha_{lj}).$$

Using these formulas and (5.2), by direct computation, we can finally obtain (5.5) and (5.6). Furthermore, since

$$(J\Delta\mathbf{A})_{ij} = -\Delta h^{n+2}_{ij}\nu_{n+1} + \Delta h^{n+1}_{ij}\nu_{n+2},$$

(5.7) follows easily from (5.5) and (5.6). □

Lemma 5.4. *Along the perturbed SMCF (1.8), we have*

$$\frac{\partial}{\partial t}|\boldsymbol{A}|^2 = \varepsilon\Delta|\boldsymbol{A}|^2 - 2\epsilon|\nabla\boldsymbol{A}|^2 + 2\langle J\Delta\boldsymbol{A}, \boldsymbol{A}\rangle + \boldsymbol{A}*\boldsymbol{A}*\boldsymbol{A}*\boldsymbol{A}. \tag{5.8}$$

In particular, we have

$$\frac{d}{dt}\int_\Sigma |\boldsymbol{A}|^2 d\mu \le -2\epsilon\int_\Sigma |\nabla\boldsymbol{A}|^2 d\mu - \epsilon\int_\Sigma |\boldsymbol{H}|^2|\boldsymbol{A}|^2 d\mu + C(n)\int_\Sigma |\boldsymbol{A}|^4 d\mu. \tag{5.9}$$

Proof. From (5.3), we see that $\frac{\partial}{\partial t}g_{ij} = \mathbf{A}*\mathbf{A}$, which implies that $\frac{\partial}{\partial t}g^{ij} = \mathbf{A}*\mathbf{A}$. Therefore, using (5.7), we compute:

$$\begin{aligned}
\frac{\partial}{\partial t}|\mathbf{A}|^2 &= \frac{\partial}{\partial t}\left(g^{ik}g^{jl}h^\alpha_{ij}h^\alpha_{jl}\right) = \mathbf{A}*\mathbf{A}*\mathbf{A}*\mathbf{A} + 2\left\langle \mathbf{A}, \frac{\partial}{\partial t}\mathbf{A}\right\rangle \\
&= 2\varepsilon\langle\mathbf{A}, \Delta\mathbf{A}\rangle + 2\langle J\Delta\mathbf{A}, \mathbf{A}\rangle + \mathbf{A}*\mathbf{A}*\mathbf{A}*\mathbf{A} \\
&= \varepsilon\Delta|\mathbf{A}|^2 - 2\epsilon|\nabla\mathbf{A}|^2 + 2\langle J\Delta\mathbf{A}, \mathbf{A}\rangle + \mathbf{A}*\mathbf{A}*\mathbf{A}*\mathbf{A}.
\end{aligned}$$

Furthermore, from (5.8) and (5.4), we have

$$\begin{aligned}
\frac{d}{dt}\int_\Sigma |\mathbf{A}|^2 d\mu &= \int_\Sigma \left(\varepsilon\Delta|\mathbf{A}|^2 - 2\epsilon|\nabla\mathbf{A}|^2 + 2\langle J\Delta\mathbf{A}, \mathbf{A}\rangle + \mathbf{A}*\mathbf{A}*\mathbf{A}*\mathbf{A} - \varepsilon|\mathbf{H}|^2|\mathbf{A}|^2\right) d\mu \\
&= \int_\Sigma \left(-2\epsilon|\nabla\mathbf{A}|^2 - 2\langle J\nabla\mathbf{A}, \nabla\mathbf{A}\rangle + \mathbf{A}*\mathbf{A}*\mathbf{A}*\mathbf{A} - \varepsilon|\mathbf{H}|^2|\mathbf{A}|^2\right) d\mu \\
&= \int_\Sigma \left(-2\epsilon|\nabla\mathbf{A}|^2 + \mathbf{A}*\mathbf{A}*\mathbf{A}*\mathbf{A} - \varepsilon|\mathbf{H}|^2|\mathbf{A}|^2\right) d\mu \\
&\le -2\epsilon\int_\Sigma |\nabla\mathbf{A}|^2 d\mu - \epsilon\int_\Sigma |\mathbf{H}|^2|\mathbf{A}|^2 d\mu + C(n)\int_\Sigma |\mathbf{A}|^4 d\mu.
\end{aligned}$$

Here, we have used Lemma 2.3, integration by parts and the fact that the complex structure J is anti-symmetric. □

In order to get the evolution equation for the covariant derivatives of the second fundamental form, we recall the following commutation formulas (see Lemma 3.2 of [14]).

Lemma 5.5. *Suppose g_t is a family of metric on Σ satisfying $\frac{\partial g_t}{\partial t} = h$. Let Δ and ∇ be the Laplacian and connection induced by g_t. Then for any tensor S on Σ, we have*

$$\frac{\partial}{\partial t}\nabla S - \nabla\frac{\partial}{\partial t}S = S*\nabla h, \tag{5.10}$$

$$\nabla(\Delta S) - \Delta(\nabla S) = \nabla Rm * S + Rm * \nabla S. \tag{5.11}$$

Here Rm is the curvature tensor on Σ.

From (5.3), (5.7), and using Gauss equation, we can prove by induction that

Lemma 5.6. *Along the perturbed SMCF (1.8), we have for any integer $l \geq 0$,*

$$\frac{\partial}{\partial t}\nabla^l \boldsymbol{A} = \varepsilon\Delta\nabla^l \boldsymbol{A} + J\Delta\nabla^l \boldsymbol{A} + \sum_{i+j+k=l} \nabla^i \boldsymbol{A} * \nabla^j \boldsymbol{A} * \nabla^k \boldsymbol{A}. \tag{5.12}$$

As a consequence, we have

$$\begin{aligned} \frac{\partial}{\partial t}|\nabla^l \boldsymbol{A}|^2 &\leq \varepsilon\Delta|\nabla^l \boldsymbol{A}|^2 - 2\epsilon|\nabla^{l+1}\boldsymbol{A}|^2 + \left\langle J\Delta\nabla^l \boldsymbol{A}, \nabla^l \boldsymbol{A}\right\rangle \\ &\quad + c(n,l) \sum_{i+j+k=l} |\nabla^i \boldsymbol{A}| \cdot |\nabla^j \boldsymbol{A}| \cdot |\nabla^k \boldsymbol{A}| \cdot |\nabla^l \boldsymbol{A}|, \end{aligned} \tag{5.13}$$

where $c(n,l)$ is a constant depending on n and l.

Proof. By (5.3), we see that $h = \frac{\partial g}{\partial t} = \mathbf{A} * \mathbf{A}$, so that $\nabla h = \nabla \mathbf{A} * \mathbf{A}$. Inductively applying (5.10), we get that

$$\begin{aligned} \frac{\partial}{\partial t}\nabla^l \mathbf{A} &= \nabla \frac{\partial}{\partial t}\nabla^{l-1}\mathbf{A} + \nabla^{l-1}\mathbf{A} * \nabla \mathbf{A} * \mathbf{A} \\ &= \nabla\left(\nabla\frac{\partial}{\partial t}\nabla^{l-2}\mathbf{A} + \nabla^{l-2}\mathbf{A} * \nabla\mathbf{A} * \mathbf{A}\right) + \nabla^{l-1}\mathbf{A} * \nabla\mathbf{A} * \mathbf{A} \\ &= \nabla^2\frac{\partial}{\partial t}\nabla^{l-2}\mathbf{A} + \nabla^{l-1}\mathbf{A} * \nabla\mathbf{A} * \mathbf{A} + \nabla^{l-2}\mathbf{A} * \nabla^2\mathbf{A} * \mathbf{A} + \nabla^{l-2}\mathbf{A} * \nabla\mathbf{A} * \nabla\mathbf{A} \\ &= \cdots \\ &= \nabla^l \frac{\partial}{\partial t}\mathbf{A} + \sum_{i+j+k=l} \nabla^i\mathbf{A} * \nabla^j\mathbf{A} * \nabla^k\mathbf{A} \\ &= \varepsilon\nabla^l\Delta\mathbf{A} + J\nabla^l\Delta\mathbf{A} + \sum_{i+j+k=l} \nabla^i\mathbf{A} * \nabla^j\mathbf{A} * \nabla^k\mathbf{A}. \end{aligned}$$

Here in the last equality, we have used (5.7) and Lemma 2.3.

Next, note that by Gauss equation, the curvature tensor Rm on Σ can be expressed as $Rm = \mathbf{A} * \mathbf{A}$, so that $\nabla Rm = \nabla\mathbf{A} * \mathbf{A}$. Then inductively applying (5.11), we get that

$$\begin{aligned} \nabla^l\Delta\mathbf{A} &= \nabla^{l-1}\left(\Delta\nabla\mathbf{A} + \nabla\mathbf{A} * \mathbf{A} * \mathbf{A}\right) \\ &= \nabla^{l-2}\left(\Delta\nabla^2\mathbf{A} + \nabla\mathbf{A} * \nabla\mathbf{A} * \mathbf{A} + \nabla^2\mathbf{A} * \mathbf{A} * \mathbf{A}\right) + \sum_{i+j+k=l} \nabla^i\mathbf{A} * \nabla^j\mathbf{A} * \nabla^k\mathbf{A} \\ &= \nabla^{l-2}\Delta\nabla^2\mathbf{A} + \sum_{i+j+k=l} \nabla^i\mathbf{A} * \nabla^j\mathbf{A} * \nabla^k\mathbf{A} \\ &= \cdots \\ &= \Delta\nabla^l\mathbf{A} + \sum_{i+j+k=l} \nabla^i\mathbf{A} * \nabla^j\mathbf{A} * \nabla^k\mathbf{A}. \end{aligned}$$

Combining the above two equalities together gives us (5.12). Then (5.13) follows easily. □

In particular, using Lemma 5.2 and Lemma 5.6, we can easily see that

Lemma 5.7. *Along the perturbed SMCF (1.8), we have*

$$\frac{d}{dt}\int_\Sigma |\nabla^l \boldsymbol{A}|^2 d\mu \leq c(n,l) \sum_{i+j+k=l} \int_\Sigma |\nabla^i \boldsymbol{A}| \cdot |\nabla^j \boldsymbol{A}| \cdot |\nabla^k \boldsymbol{A}| \cdot |\nabla^l \boldsymbol{A}| d\mu. \tag{5.14}$$

Next, let's recall the following interpolation inequality proved by Hamilton ([13], Section 12)

Lemma 5.8. *If T is any tensor and if $1 \le i \le l-1$, then with a constant $C = C(n,l)$ depending only on $n = dim\Sigma$ and l, which is independent of the metric g and the connection Γ, we have the estimate*

$$\int_\Sigma |\nabla^i T|^{\frac{2l}{i}} d\mu \le C \max_\Sigma |T|^{2\left(\frac{l}{i}-1\right)} \int_\Sigma |\nabla^l T|^2 d\mu.$$

Finally, by using (5.14) and Lemma 5.8 in the same way as in Section 7 of [16], we can obtain that

Lemma 5.9. *Along the perturbed SMCF (1.8), we have*

$$\frac{d}{dt}\int_{\Sigma(t)} |\nabla^l \boldsymbol{A}|^2 d\mu \le c(n,l) \max_\Sigma |\boldsymbol{A}|^2 \int_\Sigma |\nabla^l \boldsymbol{A}|^2 d\mu. \tag{5.15}$$

5.3. **Proof of the main theorem.** Now we come back to the case of two dimensional SMCF in $\mathbb{R}^4$ and finish the proof of local existence of SMCF.

Proof of Theorem 1.1. By Lemma 5.1, we know that for each $\varepsilon > 0$, there exists a positive time $T_\varepsilon > 0$ and a smooth solution F_ε to (5.1) on the time interval $[0, T_\varepsilon)$. For convenience, we denote the second fundamental form of F_ε at time t by $\mathbf{A}_\varepsilon(t)$.

For any $0 < \varepsilon < 1/4$, define a time $T_\varepsilon^{'} \le T_\varepsilon$ by

$$T_\varepsilon^{'} := \inf\{t \in [0, T_\varepsilon) : \|\mathbf{A}_\varepsilon(t)\|_{H^{2,2}} = 2\|\mathbf{A}_0\|_{H^{2,2}}\}.$$

Obviously, we have $T_\varepsilon^{'} > 0$ and by definition for each ε and $t \in [0, T_\varepsilon^{'})$, we have

$$\|\mathbf{A}_\varepsilon(t)\|_{H^{2,2}} \le 2\|\mathbf{A}_0\|_{H^{2,2}} := B.$$

Moreover, from Lemma 5.2, we know that the volume of $\Sigma_\varepsilon(t) := F_\varepsilon(t, \Sigma)$ satisfies

$$\frac{d}{dt}|\Sigma_\varepsilon| = -\varepsilon \int_{\Sigma_\varepsilon} |\mathbf{H}_\varepsilon|^2 d\mu_\varepsilon \ge -2\varepsilon \|\mathbf{A}_\varepsilon(t)\|_{L^2}^2 \ge -\frac{1}{2}B^2.$$

Thus for a fixed time $T_1 := |\Sigma_0|/B^2$, we have uniform bounds of the volume on $[0, T_1]$ given by

$$|\Sigma_0| \ge |\Sigma_\varepsilon(t)| \ge |\Sigma_0| - \frac{1}{2}B^2 T_1 = \frac{|\Sigma_0|}{2} := m. \tag{5.16}$$

Therefore, applying Theorem 4.8, we obtain a uniform C^0 bound of the second fundamental form

$$\|\mathbf{A}_\varepsilon(t)\|_{C^0} \le C(B, m) \tag{5.17}$$

on the time interval $[0, T_\varepsilon'] \cap [0, T_1]$. It follows from Lemma 5.9 and (5.17) that for any $0 \le l \le k$, we have

$$\frac{d}{dt}\int_{\Sigma_\varepsilon(t)} |\nabla^l \mathbf{A}_\varepsilon(t)|^2 d\mu \le c(2,l) C(B,m)^2 \int_{\Sigma_\varepsilon(t)} |\nabla^l \mathbf{A}_\varepsilon(t)|^2 d\mu.$$

where $c(2,l)$ is the constant given by Lemma 5.9. Consequently, by Gronwall's inequality, we obtain

$$\|\mathbf{A}_\varepsilon(t)\|_{H^{l,2}} \le e^{c_l C(B,m)^2 t} \|\mathbf{A}_0\|_{H^{l,2}}, \tag{5.18}$$

where $c_l := \max\{c(2,0), \cdots, c(2,l)\}$ only depends on l.

Now if $T_\varepsilon' < T_1$, then by letting $t \to T_\varepsilon^{'}$ in (5.18), we find

$$2\|\mathbf{A}(0)\|_{H^{2,2}} \le e^{c_2 C(B,m)^2 T_\varepsilon^{'}} \|\mathbf{A}_0\|_{H^{2,2}}.$$

It follows that

$$T_\varepsilon^{'} \ge \frac{\log 2}{c_2 C(B,m)^2} := T_2.$$

Thus we get a uniform lower bound for $T_\varepsilon^{'}$ given by $T_0 := \min\{T_1, T_2\}$, which is decided by $\|\mathbf{A}_0\|_{H^{2,2}}$ and $|\Sigma_0|$.

Next we restrict ourselves on the time span $[0,T_0]$. For any $\varepsilon\in(0,1/4)$, we have uniform bounds of the volume $|\Sigma_\varepsilon(t)|$ by (5.16) and the C^0-norm of $\mathbf{A}_\varepsilon(t)$ by (5.17). Moreover, if $F_0\in C^\infty$ and hence $\mathbf{A}_0\in C^\infty$, we have uniform bounds on $H^{l,2}$-norm of $\mathbf{A}_\varepsilon(t)$ for any $l\geq 0$ by (5.18). Then Lemma 5.8 yields

$$\int_{\Sigma_\varepsilon(t)}|\nabla^k\mathbf{A}_\varepsilon(t)|^p d\mu\leq C(k,p),$$

for all $k\geq 0$ and $p>0$. Then applying a version of Michael-Simon inequality (see, for example, Theorem 5.6 of [20]), we obtain that

$$\|\mathbf{A}_\varepsilon(t)\|_{C^k}\leq C(k), \tag{5.19}$$

for any $k\geq 0$.

It follows by (5.19) and standard arguments (cf. [20], Section 4) that in every local charts, we have

$$\|\partial^k F_\varepsilon(t)\|_\infty,\ \|\partial^k\partial_t F_\varepsilon(t)\|_\infty\leq C(k,F_0),$$

for any $k\geq 0$, where ∂ is the partial derivatives in the local charts. Then by Arzela-Ascoli Theorem, we conclude that there is a sub-sequence F_{ε_i} converging smoothly to a limit $F_\infty\in C^\infty([0,T_0]\times\Sigma)$ for some sequence ε_i with $\lim_{i\to\infty}\varepsilon_i\to 0$.

Finally, by taking $\varepsilon_i\to 0$ in (5.1), it easy to verity that F_∞ is a smooth solution to the SMCF (1.7). □

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